DIFFERENCE METHODS FOR HYPERBOLIC EQUATIONS

Tatsuo NOGI

June 1971

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by Tatsuo Nogi

Department of Applied Mathematics and Physics
Faculty of Engineering
Kyoto University
Kyoto, Japan

1971. 6.

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Foreword

Here we will give a short history of the method and the theory of difference scheme for the problems of partial differential equations, especially of hyperbolic equations and tell our contribution to them. In 1928 Courant, Friedrichs and Lewy wrote the paper "On the Partial Difference Equations of Mathematical Physics" which has formed the basis of modern investigations into the numerical analysis of partial differential equations. At the time they were interested in difference equations as a tool for proving the existence of solutions of partial differential equations.*) Here the portion of the paper that was devoted to hyperbolic difference equations is

*) See the inside of the back cover.

reviewed in terms of its basic contribution to the numerical solution of partial differential equations: The authors start by pointing out that a scheme cannot be a convergent one if the ratio of the time step and the space mesh width is so large that the domain of dependence of a point in the difference scheme does not contain all points of its domain of dependence in the differential equation; such a scheme ignores information which does influence the value of the solution of the differential equation." The authors constructed a centered difference scheme for the wave equation and proved its convergence by carrying over the energy method. "The crux of that method is a quadratic identity for their scheme obtained as follows : multiply the difference equation by the centered approximation to $\frac{\partial u}{\partial t}$, write the resulting quadratic function as a divergence sum over a tetrahedral region bounded by diagonal planes and observe that if the faces of the tetrahedron slope steeper than the chracteristic cone, then the surface sums over them are positive." "Once the energy identities are derived, they prove convergence swiftly and elegantly." But they gave no error estimate ; "of course, using the energy inequality the authors could have estimated the difference between exact and approximate solution with the greatest of ease, had they bothered." The method of C.-F.-L. is applied also in our process of getting the energy inequalities in Part IV .

The swift and dramatic rise of the computer during and after

World War II as a new mathematical aid in every sort of human endeavor has, as one might expect, been reflected in the field of numerical analysis. It is known that at the time J. von Neumann was convinced that numerical computation by high speed electronic computer does itself give essential answers to many complicated and unsolved problems in science and expounded it to many peoples in various branches. And at the end of the 30's he studied himself the theoretical problem in fluid mechanics, especially methods solving partial differential equations analitically unsolved. After the War he participated in meteorology which has been expected to develop greatly by numerical computation, and farther spended some time in helping computation of the continually extended problems in nuclear physics. During those times he obtained the same condition of the "mesh ratio" from a study of error-growth (stability of the difference scheme) as Courant, Friedrichs and Lewy did from a study of convergence. The partly heuristic technique of stability analysis developed by von Neumann was applied by him to a wide variety of difference and differential equation problems. This method was very briefly mentioned in his literature and a detailed discussion was not yet published. O'Brien, Hyman and Kaplan made such a discussion and numerical experiment. Later the notion of stability turned out to be very important theoretically as well as practically.

Before discussing the theory of stability we will mention some works on numerical computation of the fundamental equation of compressible and inviscid fluid mechanics. It is well known that such anequation has generally a solution having discontinuities as we call shock waves. When such a solution is demanded numerically, we need some device. In the methods used well up to the time, they solve the equation in a region having smooth solution by the method of characteristics and connect solutions on both sides of a shock by the Rankine-Hugoniot's condition. However a shock wave moves against the fixed net in space, the Rankine-Hugonit's condition is nonlinear and moreover the position of a shock wave is unknown beforehand, so that an algorithm of such a method is much complicated. Therefore it was hoped that we can compute through by a difference scheme without considering the position of a shock wave. When a shock wave arises, of course, special device are needed in the difference scheme. Otherwise the discontinuity of a shock wave brings about oscillation in the solution which grows to overflow. (Instability.) Then von Neumann and Richtmyer proposed a method of solving a modified equation with an artificial nonlinear dissipation term which makes a shock wave smooth, instead of solving the original equation of fluid mechanics. Friedrichs proposed a simple difference scheme containing a linear dissipation term and Lax applied it to nonlinear equations of fluid mechanics. Furthermore Lax and Wendroff constructed an efficient scheme with accuracy of the second order containing a quadratic dissipation term which assures a fairly narrow shock width. However Lax--Wendroff's scheme needs matrix multiplication which is troblesome for computation. In order to avoid it Richtmyer and Morton or Lapidus presented a procedure, the two-step Lax-Wendroff procedure which also has second-order accuracy. We have almost no stability theorems of these schemes for nonlinear equations, while we often check stability roughly by von Neumann's condition for the linearized equation.

But it turned out that von Neumann's condition can not prevent a "nonlinear instability", while it assures stability for linear equations.practically. First Phillips found such a fact, and Richtmyer and Morton considered a simple nonlinear hyperbolic equation and proved that its leap-frog difference scheme shows a nonlinear instability for a special perturbation of a solution. In order to show that there is an instability essential for nonlinear equations and it can not be checked by a simple extension of linear stability analysis, we also give an example with nonlinear-instability feature. In Part I we show the existence of an initial data with which the solution of the two-step Lax-Wendroff's difference equation(or the Lax--Wendroff-Richtmyer's equations) for the same simple hyperbolic equation diverges from the corresponding exact solution of the differential equation under even so small mesh ratio λ that the von Neumann's condition is satisfied by linearized analysis.

Also in U.S.S.R. computational methods for the hyperbolic equation of compressible fluid mechanics have been studied in 1960's. Godunov proposed an excellent method which has a clear physical meaning, in fact which takes account of decay of discontinuity at each mesh and formally has dissipative terms analogous to the Lax-Wendroff's scheme. Recently we tried to compare such schemes by numerical computation for a Riemann problem of the equation of one-dimensional compressible fluid mechanics, the results of which is reported in our book "The foundation of numerical analysis (1969)". ([4])

On the other hand, the stability theory of linear difference schemes has been develoved during the last two decades.

"For simplicity we shall discuss first-order systems and one-level difference operators associated with them. In a one-level scheme the value of the approximate solution at $t+\Delta t$ is obtained by applying an operator S_h to its values at time t. ($\Delta t=\lambda h$, λ=const.). S_h' is a difference operator of the form $S_h=\sum a_\alpha T^\alpha$, α a multi-index, T^α translation by $h\alpha$, and a_α a matrix-valued function. The operators S_h act on vector-valued functions normed by the L^2 norm. In the operator language, the stability of a scheme is defined by the uniform boundedness of powers of the operator S_h, i.e., the existence of two constants A and B such that

$$\|S_h^n\| \leqslant Ae^{Bt}, \text{ where, } t=n\Delta t$$

the norm being the L^2 operator norm." "The accuracy of a difference scheme is measured by how closely solutions of the differential equation satisfy the difference equation. A scheme is called accurate of order m if for every smooth solution u of the differential equation there is a constant C such that

$$\|u(t+\Delta t) - S_h u(t)\| \leqslant Ch^{m+1} \text{ ."}$$

Lax proved that for a scheme with accuracy of order $m \geqslant 1$ the stability is the necessary and sufficient condition for convergence in the sense of

$$\|u(t) - S_h^n u(o)\| \to 0 \quad (\text{as } h \to o) \ , \quad t=n\Delta t \ ,$$

for smooth data u(o). (Lax's equivalence theorem.) Furthermore it can be easily shown that for a stable scheme with accuracy of order m the overall error is $O(h^m)$:

$$\|u(t) - S_h^n u(o)\| \leqslant De^{Et} h^m \ , \quad t=n\Delta t \ .$$

Thus the problems of stabibily turns out to be most important in the theory of difference schemes." The symbol (amplification matrix) of the operator S_h is

$$S(x,\xi) = \sum_\alpha a_\alpha(x) e^{i\alpha\xi} \ .$$

For operator with coefficients independent of x it follows immediately from the isometric character of Fourier transformation that neccessary and sufficient for stability is the uniform boundedness of all powers of the symbol. This reduces a stability question to a pure matrix problem. An obvious necessary condition, the above mentioned one due to von Neumann, is for the spectrum of such a matrix to lie in the unit disc in the complex plane. An obvious sufficient condition is for the norm of this matrix to be $\leqslant 1$. A non-obvious sufficient condition is that the numerical range of the matrix be in the unit disc. Necessary and sufficient conditions were given by Kreiss, Buchanan, Morton and Schechter." (see [4])

"It is not hard to show that the von Neumann condition is necessary for the stability of schemes with variable coefficient as well. Von Neumann conjectured that it is also sufficient ; something like has been demonstrated by a surprisingly elaborate extension of the energy method." "It is possible to derive energy inequalities as was done by Lax and Wendroff with the aid of a certain amount of dissipation, and by Lax and Nirenberg with the aid of nothing at all except sufficient differentiability of the coefficients. The theorem of Lax and Nirenberg reads : If the symbol is a sufficiently differentiable function of x, ξ and if $|S(x,\xi)| \leqslant 1$ for all x and ξ, then the operator S_h satisfies the inequality

$$(*) \qquad \|S_h\| \leqslant 1+Kh .$$

Obviously the last expression implies stability in any finite time interval. The more delicate problem of proving stability

when the symbol does not satisfy the inequality $|S(x,\xi)| \leqslant 1$ but merely satisfies the von Neumann condition has been handled by Kreiss." He introduced a notion of dissipation in the following sense : the spectral radius $r(x,\xi)$ of the symbol satisfies the inequality $|r(x,\xi)| \leqslant 1 - \delta|\xi|^{2p}$ for all x, ξ , and integer p and a constant δ . "And he found that if in addition to the von Neumann condition one requires such dissipation then it is possible to introduce a new norm equivalent to the L^2 norm for which inequality (*) is true. The proof of the surprisingly delicate matrix theorems needed to do this have been simplified by Parlett.". These theories for systems with variable coefficients were investigated mainly for symmetric hyperbolic systems.
But there is an important non-symmetric hyperbolic system which we call a regularly hyperbolic system. We found that there are a kind of stable difference schemes which hold the same matrix structure as in the original hyperbolic system, in which, for example, are contained the crude but useful method introduced by Friedrichs and the more accurate method devised by Lax-Wendroff-Richtmyer. For showing that we introduced an algebra of pseudo difference operators analogous to that of pseudo differential operators considered first by Calderon and Zygmund. The essential part of the proof is given for the Friedrichs' scheme in Part II.
"Almost all problems of practical importance involve boundary conditions as well as initial conditions. An analog of the von Neumann condition has been given by Godunov and Ryabenkii using

some observations of Gelfand. Strang has found a relation to the theory of Wiener-Hopf equations and Osher has given a wide class of stable schemes using the theory of Klein-Gohberg about an integral equation with kernels depending on the difference of arguments. Kreiss has given a general theory about dissipative difference schemes for mixed problems as well. However in these theories it is supposed that the boundary is non-characteristic, and there is as yet no general theory for problems with characteristic boundaries. In order to find efficient methods for such problems we considered a piston problem as an example, we tried several numerical experiments and found a nice method. Here the piston problem is a simple model in hydrodynamics of gun-tunnels, free piston shock tubes, etc. It has been so difficult to be solved analytically that we cannot help to rely upon approximate methods. Several ones in the past depend mainly on characteristics and shock condition , so that they are very inconvenient. Thus it is necessary and is our aim to find direct algorithm solving the equation of hydrodynamics by difference schemes. Our problem itself contains much mathematical difficulty because it is not only a characteristic boundary value problem but also it has an internal moving boundary. Our method is an extension of Godunov's idea told above. This work is developed in Part III .

The above statement concerns mainly with hyperbolic systems of differential equations of the first order. But we are interested also in the mixed initial-boundary value problems of hyperbolic equations of the second order which arise in mathematical physics. These problems stand on the line of those treated by Courant, Friedrichs and Lewy who considered the initial value problems of hyperbolic equations. Ladyzenskaya has solved the mixed problems with the Dirichlet's boundary condition for the general hyperbolic equation of the second order with variable coefficients by a difference scheme. But as far as we know, there has been no simple difference scheme for solving the mixed problems with the Neumann's boundary condition or the boundary condition of the third kind. Then we proceeded to finding good schemes for those problems. For clearing the position of our investigation we will first see the works about difference schemes of boundary value problems of the equations of the second order appearing in mathematical physics.

For the Dirichlet's problem of elliptic equations, for example, the Laplace equation, Courant, Friedrichs and Lewy gave a usual difference approximation and proved its convergence by considering a minimum problem of corresponding quadratic forms. Gershgorin showed the rate of convergence of the C.F.L. schemes by the majorant method, and Collatz improved it by interpolation of boundary values. In these estimates the magnitudes of the higher-order-derivatives of the solution itself are contained, while the estimates using only the data were given later by Volkov, Bahvalov, Bramble and Hubbard, etc. On the other hand, for the problems with the Neumann's condition or the condition of the third kind for the Laplace equation Batchlet gave an algorithm in which normals through some numbers of boundary points are drawed on the plane covered by square mesh, and the function values on the intersecting points of the normals with the boundary and the mesh lines are counted as unknown values in the algebraic equation to be solved. Avoiding such difficulty of boundary approximation, Lebedev has given a simple scheme relying on the Green's formula and has proven weak convergence in $W_2^{(1)}$.

For such elliptic problems the finite element method, so called, is very efficient and has grown up to be used often lately. In this method a reduced minimum problem from an original boundary value problem is solved approximately in a subspace spanned by a class of finite number of "element" functions and their translated functions, and then the equation and the boundary condition are approximated naturally by the resulting algebraic systems. Courant proposed such idea first, and we have now the works of Demjanovič, Friedrichs and Keller, Oganesyan Zlamal, etc.

But there were few works about difference methods for the mixed initial and boundary value problems of hyperbolic and parabolic equations with derivative boundary conditions in a domain of any shape. Only a penalty method considered by Lions is known to be a unified one, but it is not so good approximation to be used in practical aim. Such circumstances led us to construct a useful difference scheme for mixed problems of wave equation and heat equation with boundary condition of the third kind (and also for boundary value problems of elliptic equations) in an arbitrary region on the plane. Here we use integral formulae of differential equations to construct the scheme, and such idea is taken from consideration about practical boundary approximation in our piston problems in Part Ⅲ. Construction and convergence of our schemes are reported in detail in Part Ⅳ.

In conclusion I should like to express my sincere gratitude to Prof. K.Okugawa and Prof. M.Yamaguti who have given helpful suggestion and successive encouragement, and also to my fellow who have supported me in various ways. I am indebted to Miss M.Kawamura for typewriting my manuscript.

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Part I

Nonlinear instability of the Lax-Wendroff-Richtmyer's scheme

Phillips has given an example [1] of a non-linear partial differential equation, a difference scheme for it that is stable for the corresponding linearized equation, and an exact solution of the non-linear difference scheme which explodes as $\exp\{\text{const.}/\Delta t\}$ when $\Delta t \to o$ for fixed t. Richtmyer and Morton has given a similar example in [2], where the simple hyperbolic equation

$$(1) \qquad \frac{\partial}{\partial t} u + \frac{\partial}{\partial x} \frac{u^2}{2} = 0 \quad , \quad u = u(t,x) \, ,$$

is considered, together with the leap-frog difference scheme

$$(2) \qquad u_j^{n+1} - u_j^{n-1} = \frac{\lambda}{2}\left[(u_{j+1}^n)^2 - (u_{j-1}^n)^2\right] ,$$

where $\lambda = \frac{\Delta t}{\Delta x}$, and u_j^n denotes $u(n\Delta t, j\Delta x)$ etc. It was shown that a special perturbation of a solution causes instability under the linear stability condition. In [2] numerical experiments also show that the leap-frog scheme is unstable, whereas the Lax-Wendroff-Richtmyer's scheme

$$(3) \qquad u_j^{n+1} = u_j^n - \frac{\lambda}{2}\left[(w_j^+)^2 - (w_j^-)^2\right] ,$$

$$(4) \qquad w_j^+ = \frac{1}{2}(u_{j+1}^n + u_j^n) - \frac{\lambda}{2}\left[(u_{j+1})^2 - (u_j)^2\right] ,$$

$$(5) \qquad w_j^- = \frac{1}{2}(u_j^n + u_{j-1}^n) - \frac{\lambda}{2}\left[(u_j^n)^2 - (u_{j-1}^n)^2\right]$$

gives a stable feature.

Here we show that there is an initial data with which the solution of the Lax-Wendroff-Richtmyer's difference equation diverges from the corresponding exact solution of the differential equation under any small mesh ratio λ. As an initial data we take

$$(6) \qquad u_j^0 = \begin{cases} u_0 (= \text{constant} > 0) & (j=-1,-2,\cdots) \\ -u_0 (= \text{constant} < 0) & (j=0,1,2,\cdots) \end{cases} .$$

Then for any $n \geq 0$ and j we have

$$(7) \qquad u_j^n = -u_{-j-1}^n .$$

In fact it is trivial when $n=0$ and if we suppose that it holds at n we have from (3)

$$u_{-j-1}^{n+1} = u_{-j-1}^n - \frac{\lambda}{2}\left[(w_{-j-1}^+)^2 - (w_{-j-1}^-)^2\right].$$

Here from (7)

$$\begin{aligned} w_{-j-1}^+ &= \frac{1}{2}(u_{-j}^n + u_{-j-1}^n) - \frac{\lambda}{2}\left[(u_{-j}^n)^2 - (u_{-j-1}^n)^2\right] \\ &= -\frac{1}{2}(u_{j-1}^n + u_j^n) - \frac{\lambda}{2}\left[(u_{j-1}^n)^2 - (u_j^n)^2\right] \\ &= -w_j^- \quad , \end{aligned}$$

and similarly $w^-_{-j-1} = -w^+_j$, hence

$$u^{n+1}_{-j-1} = -u_j - \frac{\lambda}{2}\left[(w^-_j)^2 - (w^+_j)^2\right] = -u^{n+1}_j .$$

Thus (7) holds for $n=0,1,2,\cdots$. From (3) we have for $j=0$,

$$u^{n+1}_0 = u^n_0 - \frac{\lambda}{2}\left[(w^+_0)^2 - (w^-_0)^2\right].$$

Noting that by (7) with $j=0$,

$$w^-_0 = \frac{1}{2}(u^n_0 + u^n_{-1}) - \frac{1}{2}\left[(u^n_0)^2 - (u^n_{-1})^2\right] = 0 ,$$

we have

(8) $$u^{n+1}_0 = u^n_0 - \frac{\lambda}{2}(w^+_0)^2.$$

Hence u^n_0 is monotone decreasing with n. And from (3) with $j=1$, we have

(9) $$u^{n+1}_1 = u^n_1 - \frac{\lambda}{2}\left[(w^+_1)^2 - (w^-_1)^2\right] .$$

Adding the last two equations,we have

$$u^{n+1}_1 + u^{n+1}_0 = u^n_1 + u^n_0 - \frac{\lambda}{2}\left[(w^+_1)^2 - (w^-_1)^2 + (w^+_0)^2\right] .$$

However from (4) and (5) the relations

(10) $$w^+_j = w^-_{j+1} , \qquad j=0,\pm 1,\pm 2,\cdots$$

hold. Thus using (9) with $j=0$, we have

$$u_1^{n+1} + u_0^{n+1} = u_1^n + u_0^n - \frac{\lambda}{2}(w_1^+)^2$$

which shows that the sum $u_1^n + u_0^n$ is also monotone decreasing with n. Generally by using (8) and (10) the relation

(11) $$\sum_{j=0}^{N} u_j^{n+1} = \sum_{j=0}^{N} u_j^n - \frac{\lambda}{2}(w_N^+)^2$$

holds for any N and the sum $\sum_{j=0}^{N} u_j^n$ is monotone decreasing. Also we see that

(12) $$\sum_{j=0}^{N} u_j^n \leq \sum_{j=0}^{N} u_j^0 = -(N+1)u_0 \qquad (u_0 > 0).$$

Thus we can prove

<u>Proposition</u> $u_0^n \to -\infty \quad (n \to \infty)$.

<u>Proof</u> Now we suppose that $\{u_0^n\}$ is bounded. Then by the monotonicity of $\{u_0^n\}$ there is a limit m_0 such that

(13) $$u_0^n \to m_0 \quad (n \to \infty) \quad (m_0 < 0) .$$

Hence by (8) we have

(14) $$w_0^+ = \frac{1}{2}(u_1^n + u_0^n) - \frac{\lambda}{2}\left[(u_1^n)^2 - (u_0^n)^2\right] \to 0 \quad (n \to \infty).$$

Thus $\{u_1^n\}$ is bounded, and $\{u_0^n + u_1^n\}$ is also bounded. The sum is monotone decreasing and has a limit. By this fact and (13) there is a limit m_1 such that

$$(15) \qquad u_1^n \to m_1 \quad (n \to \infty) .$$

From (13), (14) and (15) we have

$$(16) \qquad (m_1 + m_0)\{1 - \lambda(m_1 - m_0)\} = 0 .$$

Moreover from (14) and (10) with $j=0$

$$w_1^- \to 0 \quad (n \to \infty) .$$

Thus from (9) we have

$$(17) \qquad w_1^+ = \frac{1}{2}(u_2^n + u_1^n) - \frac{\lambda}{2}\{(u_2^n)^2 - (u_1^n)^2\} \to 0 .$$

Hence $\{u_2^n\}$ is bounded and the sum $\{u_2^n + u_1^n + u_0^n\}$ is also bounded (and it is monotone decreasing), and it has a limit. Consequently $\{u_2^n\}$ has also a limit m_2 :

$$(18) \qquad u_2^n \to m_2 \quad (n \to \infty).$$

From (15), (17) and (18) the relation

$$(m_2 + m_1)\{1 - \lambda(m_2 - m_1)\} = 0$$

holds. By similar deduction it is known that there is a limit

m_j of the sequence $\{u_j^n\}$ for each $j=0,1,2,\cdots$ and

(19) $$(m_{j+1}+m_j)\{1-\lambda(m_{j+1}-m_j)\}=0 \quad (j=0,1,2,\cdots)$$

hold. Hence we have

(20) $$m_{j+1}=-m_j \qquad \text{or} \qquad m_{j+1}=m_j+\frac{1}{\lambda} .$$

On the other hand by taking $n\to\infty$ in (12) we have

(21) $$\sum_{j=0}^{N} m_j \leqslant -(N+1)u_0 .$$

Now we will prove that (20) conflicts with (21). For the purpose the following lemma is sufficient.

<u>Lemma</u> If the sequence m_j $(j=0,1,2,\cdots)$ satisfies the condition (20), then the inequality

(22) $$\sum_{j=0}^{N} m_j \geq ([-\lambda m_0]+1)m_0$$

holds for any N, where the bracket [ ] is Gauss' symbol.

Proof of the lemma. Now we suppose that there is a positive integer j_k with $k\geqslant 0$ such that

(23) $$m_{j_k}=m_0+\frac{k}{\lambda}=\max_j\left\{m_j;\ m_j<0,\ 0\leqslant j\leqslant j_k\right\}$$

and $m_j\neq m_{j_k}$ $(j<j_k)$, and the inequality

(24) $$\sum_{j=0}^{j_k} m_j \geqslant m_0+(m_0+\frac{1}{\lambda})+ \quad +(m_0+\frac{k}{\lambda})$$

holds. If we have first at $j=j_{k+1}>j_k$

(25) $$m_{j_{k+1}} = m_0+\frac{k+1}{\lambda} = \max_j\left\{m_j;\ m_j\leqslant 0,\ 0\leqslant j\leqslant j_{k+1}\right\},$$

then for $j_{k+1}>j_k+1$

(26) $$\sum_{j=j_k+1}^{j_{k+1}-1} m_j=0 \quad ,$$

(which will be proved later.). Thus by adding (24),(25) and (26) we have

(27) $$\sum_{j=0}^{j_{k+1}} m_j \geqslant m_0+(m_0+\frac{1}{\lambda})+\cdots \quad +(m_0+\frac{k+1}{\lambda}) \quad ,$$

which holds also when $j_{k+1}=j_k+1$. Hence in general for k satisfying

(28) $$m_0+\frac{k}{\lambda}\leqslant 0,$$

we have

(29) $$\sum_{j=0}^{j_k} m_j \geqslant m_0+(m_0+\frac{1}{\lambda})+\cdots +(m_0+\frac{k}{\lambda}) \geqslant (k+1)m_0 \quad ,$$

if there is an m_{j_k} defined by (23). Such k satisfying (28) is finite and at most equal to $[-\lambda m_0]$.

Thus there is a positive integer k_0 such that we have an $m_{j_{k_0}}$ defined by (23) but no m_{j_k} for $k>k_0$.

It is the case in which either there occur only the first cases of (20) for $j \geqslant j_{k_0}$ or $m_0 + \frac{k_0}{\lambda} \leqslant 0$ but $m_0 + \frac{k_0+1}{\lambda} > 0$ (if there is an m_j such that $m_j = m_0 + \frac{k_0+1}{\lambda}$). Then we have

$$(30) \qquad \sum_{j=j_{k_0}+1}^{N} m_j \geqq 0 ,$$

for any integer $N \geqq j_{k_0}+1$ (which will be proved later).

From (29) and (30) the inequality

$$\sum_{j=0}^{N} m_j \geqq (k_0+1)m_0$$

holds for any $N \geqq j_{k_0}+1$. However as shown above $k_0 \leqq [-\lambda m_0]$, $m_0 < 0$. Consequently we have

$$(22) \qquad \sum_{j=0}^{N} m_j \geqq ([-\lambda m_0]+1)m_0 .$$

<u>Proof of (26).</u> In the sequence

$$(31) \qquad m_{j_k+1}, m_{j_k+2}, \dots, m_{j_{k+1}-1},$$

there occurrs one of (20) between two adjacent terms. And we have $m_{j_{k+1}} = m_0 + \frac{k+1}{\lambda}$ first at $j = j_{k+1}$. From these facts it is necessary that

$$(32) \qquad |m_j| \geqq |m_{j_k}| \qquad (j_k+1 \leqq j \leqq j_{k+1}-1) .$$

Thus we have

(33) $$m_{j_{K+1}-1} = m_{j_K} = -m_{j_K+1} .$$

When in the sequence (31) there occurs the phase as following

(34) $$\cdots m, -m, m, \cdots$$

(that is, there happens the former case of (20) twice successively), even if the later two term $-m$, m are omitted, we have the same value of the sum

(35) $$\sum_{j=j_K+1}^{j_{K+1}-1} m_j .$$

We put the sequence which is constructed by omitting all the concerned pairs successively from the head of the original sequence (31) in the form

(36) $$m_{j_K+1} = m^{(1)}, m^{(2)}, \cdots , m^{(p)} = m_{j_{K+1}-1} .$$

Then there are the same relations between two adjoining terms of (36) as in (31) :

(37) $$m^{(j+1)} = -m^{(j)} \quad \text{or} \quad m^{(j+1)} = m^{(j)} + \frac{1}{\lambda}.$$

But in the modified sequence such an order as (34) never happen. When the former case of (37) holds we call, for brevity, that ⊗ occurs from (j) to (j+1), or simply ⊗ occurs. When the later case of (37) holds we call that ⊕ occurs. Then we can say that ⊗ never occur twice successively. If ⊗ occurs from (1) to (2), then ⊕

occurs from (2) to (3). Therefore we must put $m^{(2)}=m_{j_{K+1}-1}$, so that we have (26) (i.e. $m^{(1)}+m^{(2)}=0$). Next we consider the case that $\oplus$ occurs from (1) to (2), $\otimes$ occurs first from (r) to (r+1) and $\oplus$ occur n times successively from (r+1). Then it must hold that $n \leqslant r-1$. Thus the total sum of the (n+1) forward terms and the (n+1) backward terms to the central $\otimes$

$$m^{(r-n)} \oplus \cdots \oplus m^{(r-1)} \oplus m^{(r)} \otimes m^{(r+1)} \oplus m^{(r+2)} \oplus \cdots \oplus m^{(r+n+1)}$$

is equal to zero. Hence we can omit these terms from the sequence (36) without varying the value of the sum (35). And we have $m^{(r-n)}=m^{(r+n+2)}$. If we write the sequence got by omitting in the form of (36), then either relation of (37) holds between each two adjacent terms and $\otimes$ does not occur twice successively. Succeeding the omitting process we arrive at the final form of the sequence such that $m^{(1)}=m_{j_K+1}$, $m^{(2)}=m_{j_{K+1}-1}$ whose sum is equal to zero by (33). Thus the original sum of (36) and consequently the sum (35) are equal to zero. Q.E.D.,

<u>Proof of (30).</u> In the case that only the former equalities of (20) occur for $j \geq j_{K_0}$, that is,

$$(38) \qquad m_{j+1}=-m_j \ , \quad j \geq j_{K_0}$$

we can easily prove (30). In fact we have then

$$(39) \qquad \sum_{j=j_{K_0}+1}^{N} m_j = \begin{cases} -m_j \ \geq 0 & (N=j_{K_0}+2p-1) \\ 0 & (N=j_{K_0}+2p) \end{cases} , \quad (p=1,2,3,\cdots).$$

Next we consider the case in which there are terms in the sequence $\{m_j\}$ such that $m_{j-1} = m_0 + \frac{k}{\lambda}$ and $m_j = m_0 + \frac{k_0+1}{\lambda}$. For any interger $N > j_{k_0}+1$ we take the sum of the sequence

$$(40) \qquad m_{j_{k_0}+1},\ m_{j_{k_0}+2},\ \cdots\ ,\ m_{N-1},\ m_N\ .$$

We take a way similar to that in the proof of (26). In the case that after two-times-occurrence of ⊗ there occurs the phase of (34), we omit the later two terms, and so on. The resulting sequence is written as follows :

$$(41) \qquad m_{j_{k_0}+1} = m^{(1)},\ m^{(2)},\ \cdots\ ,\ m^{(p)} = m_N\ .$$

Then in (41) ⊗ does not occur twice successively. Let's assume that ⊗ occurs first from (r) to (r+1) and several numbers of ⊕ follow. (r=1 is also allowed.) Now we suppose that $m^{(r+n+1)} < 0$ but $m^{(r+n+2)} > 0$ (n=0 is allowed). Then ⊕ must occur n times or more before ⊗ occurs. Hence the sum of terms

$$m^{(r-n)} \oplus m^{(r-n+1)} \oplus \cdots \oplus m^{(r-1)} \oplus m^{(r)} \otimes m^{(r+1)} \oplus \cdots \oplus m^{(r+n+1)}\ ;$$

is zero, so that we can omit these terms from our sequence. Furthermore, two adjoining terms in the resulting sequence are combined by the relation (37). Such an omitting process is succeeded, and then all the negative terms $m^{(q)}$ are eliminated. Thus we have

(42) $$\sum_{j=j_{k_0}+1}^{N} m_j \geqq 0$$

(42) is trivial when $N=j_{k_0}+1$ because then $m_N > 0$. Thus (30) is completely proved.

We have shown that (24) holds for any integer $N \geqq j_{k_0}+1$. However on the way of the proof of (26) the inequality

$$\sum_{j=j_k+1}^{N} m_j \geqq 0 ,$$

for any N such that $j_k < N < j_{k+1}$, is easily shown by the method in the latter half of the proof of (30). Hence we have the relation

$$\sum_{j=0}^{N} m_j \geqq \sum_{j=0}^{j_{k+1}} m_j > \sum_{p=0}^{k+1} (m_0 + \frac{p}{\lambda}) ,$$

from which (22) is again got. Thus we have proven our lemma completely.

Remark 1. The proof of nonuniform-boundedness of the sequence $\{u_j^n\}$ is more simple, because we have the boundedness of each sum $\sum_{j=0}^{N} u_j^n (N=0,1,2,\cdots)$ immediately from the hypothesis that the sequence $\{u_j^n\}$ is uniformly bounded.

Remark 2 As we understand from the above proof, we need not limit initial values to (3). For example, if the initial values satisfy the relations

$$u_j^0 = -u_{-j-1}^0 \quad (j=0,\ 1,\ 2,\ \cdots), \quad u_j^0 < 0 \quad (j=0,1,2,\cdots),$$

and $\sum_{j=0}^{\infty} u_j^0 = -\infty$, then we have also for any $\lambda > 0$

$$u_0^n \to -\infty \quad (n \to \infty) .$$

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Part II

Stability of the Friedrichs'scheme for regularly hyperbolic systems

Introduction

In 1956 Lax and Richtmyer [1] proved that for a well posed evolutional system of partial differential equations a solution of a consistent difference analogue converges to the corresponding exact solution of the original problem if and only if the difference scheme is stable. After that many works are devoted to the investigation of stability. For pure initial value problems with constant coefficients Kreiss [2] gave several necessary and sufficient conditions for stability. For the history and the detail, see the author's M.C. thesis [13].

For the problems of the symmetric hyperbolic system of first order with variable coefficients Friedrichs [3] gave a simple stable difference scheme with accuracy of order one. Lax and Wendroff [4] proposed an efficient scheme with accuracy of order two, the stability of which was proved in the case of constant coefficients. Kreiss [5] proved by complicated technique that the Lax-Wendroff's scheme (and more general dissipative schemes) with variable coefficients is stable. Furthermore Lax and Nirenberg [6] also proved elegantly the stability of the same scheme by using the sharp Gårding's inequality.

However Yamaguti [7] showed that the Lax-Wendroff's scheme foranonsymmetric hyperbolic system is not necessarily stable. Here we will show that the Friedrichs' scheme and the modified Lax-Wendroff's scheme for regularly hyperbolic systems with variable coefficients are stable, by using a new notion of "pseudo--difference operator". This work was done in close collaboration with Prof. Yamaguti, and see also [8]. Recently Vaillancourt [9] gave a strong form of our stability theorem.

§1 The Friedrichs'scheme

Let us consider the Cauchy problem for a hyperbolic system of partial differential equations:

$$(1.1) \qquad \frac{\partial u}{\partial t} = \sum_{\ell=1}^{n} A_\ell(x) \frac{\partial u}{\partial x_\ell} .$$

Here $u=u(x,t)=(u_1,u_2,\cdots,u_m)$ is a complex valued m-vector function depending on real variables $x=(x_1,\cdots,x_n)$ and t, and $A_\ell=A_\ell(x)$ are $m\times m$ matrices depending smoothly on x (but for convenience not on t) and equal to constants for large x. **This does not lose grnerality.**

We make a difference analogue for (1.1)

$$(1.2) \qquad u(t+\Delta t)=S_h u(t),$$

where S_h is a difference operator depending on a parameter $h>o$ as the following;

$$(1.3) \qquad S_h = \frac{1}{2n} \sum_{\ell=1}^{n} \left[(I+\lambda n A_\ell) T_{x_\ell} + (I-\lambda n A_\ell) T_{x_\ell}^{-1} \right] .$$

Here $\Delta t>o$ is the time step connected with a space mesh width h by a relation $\Delta t=\lambda h$, $\lambda=\text{const.}>0$, I is the unit operator and T_{x_ℓ} and $T_{x_\ell}^{-1}$ $(\ell=1,\cdots,n)$ are the shift operators:

$$(T_{x_\ell}u)(x)=u(x_1,\cdots,x_{\ell-1},x_\ell+h,x_{\ell+1},\cdots,x_n)\ ,$$

$$(T_{x_\ell}^{-1}u)(x)=u(x_1,\cdots,x_{\ell-1},x_\ell-h,x_{\ell+1},\cdots,x_n)\ .$$

We call suchascheme the Friedrichs' one for (1.1).

§2 Regularly Hyperbolic Systems

We assume that (1.1) is a regularly hyperbolic system, that is to say, that the matrix $\sum_{\ell=1}^{n} A_\ell(x)\xi_\ell=A(x,\xi)$ has only real distinct eigenvalues for all real $\xi=(\xi_1,\cdots,\xi_n)$ and every pair of eigenvalues $\mu_j(x,\xi)$ and $\mu_k(x,\xi)$ satisfiesthe following condition: there is a positive constant d such that

$$|\mu_j(x,\xi)-\mu_k(x,\xi)|\geqslant d \quad \text{for all} \quad x\in R^n,\ \xi\in S^{n-1}.$$

(2.1) $$(j\neq k),\quad j,\ k=1,\cdots,m.$$

$$S^{n-1}:\ \text{the surface of the unit sphere in } R^n.$$

Then we know that there is a nonsingular smooth matrix $N(x,\xi)$ whose determinant is apart from zero for all $x\in R^n$, $\xi\in S^{n-1}$, and which diagonalizes $A(x,\xi)$, i.e.

$$D(x,\xi)=N(x,\xi)A(x,\xi)N\ (x,\xi)$$

$$(2.2) \qquad = \begin{bmatrix} \mu_1(x,\xi) & & 0 \\ & \mu_2(x,\xi) & \\ & & \ddots \\ 0 & & \mu_m(x,\xi) \end{bmatrix}.$$

Wellposedness in L^2 of the Cauchy problem for the regularly hyperbolic system is well known under some conditions on the smoothness of the coefficients $A_\ell(x)$ ($\ell=1,\cdots,n$) (see [10]).

Now we state the stability theorem (see also Remark 2).

Theorem 1 If $\lambda < \frac{1}{\sqrt{n\mu_0}}$, then the scheme (1.2),(1.3) is stable in the sense of Lax-Richtmyer. Here $\mu_0 = \max \mu_j(x,\xi)$, $j=1,\cdots,m$, $|\xi|\leq 1$, $x\in R^n$.

Proof First we remark that the stability means for positive integer ν such that $\|S_h^\nu u(x,o)\| \leq C\|u(x,o)\|$, $\nu h\leq T$ where C is a constant independent of ν and h.

In order to show this inequality it is convenient to introduce a new norm which is equivalent to the usual L^2 norm. First we explain its construction and equivalence.

We put $h(x,\xi)=N^*(x,\xi)N(x,\xi)$, which is a strictly positive definite function for $x\in R^n$, $\xi\in S^{n-1}$ and has the following properties (which we call the $\mathcal{H}$-property),

i) $h(x,\xi)$ is homogeneous of degree zero in ξ.

ii) each $h(x,\xi)$ is independent of x for $|x|> R$; R is a fixed positive constant.

iii) $h(x,\xi)\in C^\infty(R^n_x\times(R^n_\xi-\{0\}))$.

Now we associate a one-parameter family of operators H_h with such a function $h(x,\xi)$ by the following formula:

$$(2.3) \qquad H_h u = \text{l.i.m.}\int e^{ix\xi} h(x,\sinh\xi)\hat{u}(\xi)d\xi, \quad u\in L^2_x ,$$

where $\sinh\xi=(\sinh\xi_1,\sinh\xi_2,\cdots,\sinh\xi_n)$ and $\hat{u}(\xi)=\mathcal{F}u=\text{l.i.m.}\int e^{-ix\xi}u(x)dx$
We can prove that H_h is a bounded operator in L^2_x . For that we use the following

Lemma 1 (P.D.Lax [11]) Every $h(x,\xi)$ having the $\mathcal{H}$-property can be expanded in a series

$$(2.4) \qquad h(x,\xi)= \sum_\alpha a_\alpha e^{i\alpha\frac{\xi}{|\xi|}} ,$$

α varying over all multiindices so that the series, as well as the differentiated series with respect to x or ξ, converge uniformly.

Proof of Lemma 1 Define

$$h_1(x,\xi)= \phi(|\xi|)h(x,\xi) ,$$

where $\phi(\rho)$ is an infinitely differentiable function equal to 1 at $\rho=1$ and zero outside a small interval around $\rho=1$. Regarding $h_1(x,\xi)$ as a periodic function in ξ, expand it in a Fourier series ; since $h_1(x,\xi)$ is smooth the series will converge rapidly. For $\xi=1$ the function h_1 and h are the same; thus (2.4) is obtained. Q.E.D.

Hence the operator H_h defined by (2.3) can be written in the form

$$H_h u = \sum_\alpha a_\alpha \, \ell.i.m. \int e^{ix\xi} e^{i\alpha\frac{\xi}{|\xi|}} \hat{u}(\xi) d\xi \, .$$

Therefore by the Plancherel's theorem we have

$$\|H_h u\|_{L^2} \leqslant \sum_\alpha \sup a_\alpha(x) \quad \|u\|_{L^2} \, .$$

Remark 1 We can make the algebra of one-parameter families of bounded operators H_h mapping L^2 into itself generated by $h(x,\xi)$ having the $\mathcal{H}$ -property and call it the algebra of "pseudo difference schemes".

Now we can show that for any u with a small fixed support $(Re\, H_h u, u)$ is positive definite.* In fact

$$(H_h u, u) = (H_{oh} u, u) + \sum_{\ell=1}^{n} ((x_\ell - x_{\ell 0}) H_{1\ell h} u, u) \, ,$$

H_{oh}: operator corresponding to $h(x_0, \xi)$,

$H_{1\ell h}$: operator corresponding to $h_{1\ell}(x,\xi)$ such that

$$h(x,\xi) = h(x_0,\xi) + \sum_{\ell=1}^{n} (x_\ell - x_{\ell 0}) h_{1\ell}(x,\xi) \, ,$$

$x_0 = (x_{10}, \cdots, x_{n0})$ is a point in the concerned support.

Naturally $Re(H_{oh} u, u) \geqslant d_1 \|u\|^2$. Then if we take the diameter of

* (,) means the scalar product in L^2.

that fixed support very small, we can show that (although $H_{1\ell h}$ does not belong to the operator family discussed above)

$$\left|((x_\ell - x_{\ell 0})H_{1\ell h}u,\ u)\right| \leqslant \varepsilon \|u\|^2 .$$

Therefore we get

$$\text{(2.5)} \qquad \mathrm{Re}(H_h u,\ u) \geqslant d_2\|u\|^2 .$$

We consider now a partition of unity $\{\varphi_p\}$ such that $\sum_p \varphi_p^2 = 1$. Because of the assumption about $A_\ell(x)$ $(\ell=1,\cdots,n)$, we can take a partition composed of finite number of φ_p's. If we take the maximum of diameters of the supports of φ_p (which are not contained in the region where $A_\ell(x)$ $(\ell=1,2,\cdots,m)$ are constant) sufficiently small, then the norm

$$\text{(2.6)} \qquad \|u\|_H^2 = \sum_p \mathrm{Re}(H_h\varphi_p u,\ \varphi_p u)$$

is equivalent to the L^2 norm. The inequality $\|u\|_H \leqslant C\|u\|$ is evident by the boundedness of H_h. The inverse inequality results from (2.5) by summing up the inequality (2.5) for $\varphi_p u$.

Now we can say that it suffices to prove

$$\text{(2.7)} \qquad \|S_h u\|_H \leqslant (1+O(h))\|u\|_H \quad ,$$

for the stability of (1.2). For we have then

$$\|S_h^\nu u\|_H \leqslant (1+Ch)^\nu \|u\|_H \leqslant e^{C\nu h}\|u\|_H , \qquad C=\text{const}.$$

from which by noting $\nu h \leq T$ and norm equivalency we get

$$\|S_h^{\nu} u\| \leq C_1 e^{CT} \|u\| \quad , \quad C_1=\text{const.},$$

which means stability. The inequality (2.7) can be written explicitly in the form

$$(2.8) \qquad \sum_p \operatorname{Re}(H_h \varphi_p S_h u, \ \varphi_p S_h u) \leq (1+O(h)) \sum_p \operatorname{Re}(H_h \varphi_p u, \ \varphi_p u).$$

In order to transform the left side of the last inequality, we will estimate the commutator $[\varphi_p, S_h]=\varphi_p S_h - S_h \varphi_p$. For that we note some lemmas.

We define the operator Λ_h as follows;

$$(2.9) \qquad \Lambda_h u=\text{l.i.m.}\int e^{ix\xi} |\sinh\xi| \hat{u}(\xi) d\xi \ .$$

<u>Lemma 2</u> We assume that $a(x)$ is smooth and is equal to a constant for large $|x|$. Then $a(x)\Lambda_h - \Lambda_h a(x)$ is a family of bounded operators with norm $O(h)$.

<u>Proof</u> Put $a_1(x)=a(x)-a(\infty)$. For every square integrable function u, we get

$$\mathcal{F}\left[a(x)\Lambda_h - \Lambda_h a(x)\right] u = \mathcal{F}\left[a_1(x)\Lambda_h - \Lambda_h a_1(x)\right] u$$

$$= \int \hat{a}_1(\xi-\eta) |\sinh\eta| \hat{u}(\eta) d\eta - \int \hat{a}_1(\xi-\eta) |\sinh\xi| \hat{u}(\eta) d\eta$$

$$= \int \hat{a}_1(\xi-\eta) \Big[|\sinh\eta| - |\sinh\xi| \Big] \hat{u}(\eta) d\eta \ .$$

Hence

$$\|[a(x)\Lambda_h - \Lambda_h a(x)]u\| \leqslant \|\int |\hat{a}_1(\xi-\eta)|\cdot|\sinh\xi - \sinh\eta|\cdot|u(\eta)|d\eta\|$$

$$=\|\int |\hat{a}_1(\xi-\eta)|\cdot|2\sin h\frac{(\xi-\eta)}{2}|\cdot|\frac{\cosh(\xi+\eta)}{2}|\cdot|\hat{u}(\eta)|d\eta\|$$

$$\leqslant h\|\int |\hat{a}_1(\xi-\eta)|\cdot|\xi-\eta|\cdot|\hat{u}(\eta)|\,d\eta\|$$

$$\leqslant h\|\,|\hat{a}_1(\xi)|\cdot|\xi|\,\|_{L^1}\,\|u\|\,. \qquad \text{(Q.E.D.)}$$

<u>Lemma 3</u> We assume that $a(x)$ has the same property as in Lemma2 and $k(\xi)$ has the $\mathcal{H}$-property. Denote the corresponding operator by K_h. Then we have for every $u\in L^2_x$

$$(2.10)\qquad \|\Lambda_h(a(x)K_h - K_h a(x))u\| \leqslant C\,h\,\|u\|,$$

$$(2.11)\qquad \|(a(x)K_h - K_h a(x))\Lambda_h u\| \leqslant C\,h\,\|u\|,$$

where the constant C is independent of u and h.

<u>Proof</u> $\|\Lambda_h(a(x)K_h - K_h a(x))u\|_{L^2_x}$

$$= \|\int \hat{a}(\xi-\eta)|\sin h\xi|\cdot[k(\sin h\eta) - k(\sin h\xi)]\,\hat{u}(\eta)d\eta\|_{L^2_\xi}$$

$$\leqslant \|\int \hat{a}(\xi-\eta)\big[|\sin h\xi| - |\sin h\eta|\big]\cdot k(\sin h\eta)\,u(\eta)d\eta\|_{L^2_\xi}$$

$$+\|\int \hat{a}(\xi-\eta)\big[|\sin h\eta|\,k(\sin h\eta) - |\sin h\xi|\,k(\sin h\xi)\big]\cdot\hat{u}(\eta)\,d\eta\|_{L^2_\xi}$$

$$\leqslant \|\int |\hat{a}(\xi-\eta)|\,\big||\sin h\xi| - |\sin h\eta|\big|\cdot|k(\sin h\eta)|\cdot|\hat{u}(\eta)|\,d\eta\|_{L^2_\xi} +$$

$$+\ \Big\| \int |\hat{a}(\xi-\eta)|\cdot|\sin h\xi - \sin h\eta| \times$$

$$\times \left| \frac{|\sin h\eta|\cdot k(\sin h\eta) - |\sin h\xi|\, k(\sin h\xi)}{|\sin h\xi - \sin h\eta|} \right| |\hat{u}(\eta)|\, d\eta \ \Big\| \ .$$

Hence by using the Hausdorff-Young's inequality we get

$$\| \Lambda_h (a(x)K_h - K_h a(x))u \|_{L^2_x}$$

$$\leqslant Ch \|\hat{a}(\xi)\cdot|\xi|\|_{L^1_\xi} \cdot \|u\|_{L^2_x} + Sh \|\hat{a}(\xi)\cdot|\xi|\|_{L^1_\xi} \|u\|_{L^2_x} \ ,$$

where S is a Lipshitz constant of $|\xi|\cdot k(\xi)$ for $|\xi| \leqslant 1$. The proof is to be done in the same way for $(a(x)K_h - K_h a(x))\Lambda_h u$. (Q.E.D.

Now the Friedrichs' difference operator S_h can be written in the form

$$(2.12) \qquad S_h = E_h + i\lambda Q_h \Lambda_h ,$$

where

$$(2.13) \qquad E_h u = \text{l.i.m.} \int e^{ix\xi} \sum_{\ell=1}^{n} \frac{\cos h\xi_\ell}{n} \hat{u}(\xi) d\xi \ ,$$

and Q_h is the family corresponding to $\sum_{\ell=1}^{n} A_\ell(x) \frac{\xi_\ell}{|\xi|}$ having the $\mathcal{H}$-property. Denote the families corresponding to $\frac{\xi_\ell}{|\xi|}$ (=1, ,n) by $K_{\ell h}$ $(\ell=1,\cdots, n)$ respectively. Then $Q_h = \sum_{\ell=1}^{n} A_\ell(x) K_{\ell h}$. Therefore by using Lemma 2 and Lemma 3 we have

$$\|(\varphi_p Q_h \Lambda_h - Q_h \Lambda_h \varphi_p)u\| = \|\sum_{\ell=1}^{n} (\varphi_p A_\ell K_{\ell h} \Lambda_h - A_\ell K_{\ell h} \Lambda_h \varphi_p)u\| \leqslant$$

$$\leq \sum_{\ell=1}^{n} \Big[\|A_\ell(\varphi_p K_{\ell h} - K_{\ell h}\varphi_p)\Lambda_h u\| + \|A_\ell K_{\ell h}(\varphi_p \Lambda_h - \Lambda_h \varphi_p)u\| \Big]$$

$$\leq \sum_{\ell=1}^{n} \Big[\max_x |A_\ell(x)| \cdot \|(\varphi_p K_{\ell h} - K_{\ell h}\varphi_p)\Lambda_h u\|$$

$$+ \|A_\ell K_{\ell h}\| \, \|(\varphi_p \Lambda_h - \Lambda_h \varphi_p)u\| \Big]$$

$$\leq C\, h\, \|u\| .$$

And by using the definition of E_h and the regularity of $\varphi_p(x)$, we have easily

$$\|(\varphi_p E_h - E_h \varphi_p)u\| \leq C\, h\, \|u\| .$$

Hence we get the desired estimate:

$$\|(\varphi_p S_h - S_h \varphi_p)u\| \leq \|(\varphi_p E_h - E_h \varphi_p)u\|$$

$$+ \lambda \|(\varphi_p Q_h \Lambda_h - Q_h \Lambda_h \varphi_p)u\|$$

$$\leq C\, h\, \|u\| ,$$

where the constant C depends only on the coefficients $A_\ell(x)(\ell=1,\cdots,n)$ and the partition of unity $\{\varphi_p(x)\}$. Thus

$$\Big| \sum_p \mathrm{Re}(H_h \varphi_p S_h u, \varphi_p S_h u) - \sum_p \mathrm{Re}(H_h S_h \varphi_p u, S_h \varphi_p u) \Big|$$

$$\leq \sum_p \Big[|(H_h(\varphi_p S_h - S_h \varphi_p)u, \varphi_p S_h u)| + |(H_h S_h \varphi_p u, (\varphi_p S_h - S_h \varphi_p)u)| \Big]$$

$$\leq C\, h\, \|u\|^2 .$$

Therefore for stability it suffices to prove, instead of (2.8),

$$\sum_p Re(H_h S_h \varphi_p u,\ S_h \varphi_p u) \leq (1+O(h)) \sum_p Re(H_h \varphi_p u,\ \varphi_p u) \tag{2.14}$$

Putting $v=\varphi_j u$, we will show

$$Re(H_h S_h v,\ S_h v) \leq (1+O(h)) Re(H_h v, v), \tag{2.15}$$

which means

$$Re((H_h - S_h{}^* H_h S_h)v, v) \geq -O(h)\|v\|^2, \tag{2.16}$$

where the operator $S_h{}^*$ adjoint to S_h can be written in the form

$$S_h{}^* = E_h{}^* - i\lambda \Lambda_h Q^*, \quad Q_h{}^* \text{ is adjoint to } Q_h. \tag{2.17}$$

Now we put $P_h = H_h - S_h{}^* H_h S_h$ and we get, using (2.17),

$$P_h = P_h^{(0)} + i\lambda P_h^{(1)} + \lambda^2 P_h^{(2)}, \tag{2.18}$$

where $P_h^{(0)} = H_h - E_h{}^* H_h E_h$,

$$P_h^{(1)} = \Lambda_h Q_h{}^* H_h E_h - E_h{}^* H_h Q_h \Lambda_h, \tag{2.19}$$

$$P_h^{(2)} = \Lambda_h Q_h{}^* H_h Q_h \Lambda_h,$$

Here we note three important lemmas :

Lemma 4 If K_h is a family associated with k having the $\mathcal{H}$-property, then for every $u \in L^2_x$,

$$(2.20) \qquad \|(\Lambda_h K_h - K_h \Lambda_h)u\| \leqslant O(h)\|u\| .$$

Proof First we remark that if $\|\Lambda_h K_h^{(n)} - K_h^{(n)}\Lambda_h\| \leqslant Ch$ with C independent of n, and if $K_h^{(n)} \to K_h$ $(n \to +\infty)$ with respect to the operator norm, then $\|\Lambda_h K_h - K_h \Lambda_h\| \leqslant Ch$. On the other hand from Lemma 1 we see that the follwing finite sums of special kernels are dense in the class of functions having the $\mathcal{H}$-property with respect to the topology of $C^1(R^n_x \times (R^n_\xi - \{o\}))$:

$$\sum_\alpha^{\text{finite}} a_\alpha(x)k_\alpha(\xi), \qquad k_\alpha(\xi),\ a_\alpha(x) \text{ having the } \mathcal{H}\text{-property.}$$

Here we can assume that $k_\alpha(\xi)$ are scalar according to the Lemma 1. Therefore it suffices to prove Lemma 4 for the case of K_h associated with the finite sum:

$$(2.21) \qquad k\ (x,\xi) = \sum_\alpha^{\text{finite}} a_\alpha(x)k_\alpha(\xi).$$

We have already

$$\| a_\alpha(x)\Lambda_h - \Lambda_h a(x) \| \leqslant h \| \hat{a}_{1\alpha}(\xi)\,|\xi|\,\|_L ,$$

but we also have

$$\| \hat{a}_\alpha(\xi)\,|\xi|\,\|_{L^1} \leqslant \frac{M}{(1+|\alpha|)^k} ,$$

for some constant M because by using the fact that the support of $a_{1\alpha}(x)$ are contained in a fixed compact,

$$\|\hat{a}_\alpha(\xi)|\xi|\|_{L^1} \leqslant \left(\int \frac{d\xi}{(1+|\xi|)^{2n}} \quad (1+|\xi|)^{2(n+1)} |\hat{a}_\alpha(\xi)|^2 \, d\xi\right)^{\frac{1}{2}}$$

$$\leqslant M \sup_{\substack{x \\ |\beta| \leqslant n+1}} \left| D_x^\beta a_\alpha(x) \right| \leqslant \frac{M}{(1+|\alpha|)^k} \; .$$

Here we can take k large enough since $a_{1\alpha}(x) \in C_0^\infty$. Then we get

$$\|\Lambda_h K_h^{(n)} - K_h^{(n)} \Lambda_h\| \leqslant \sum_\alpha \|\Lambda_h a_\alpha - a_\alpha \Lambda_h\| \; \|K_{\alpha h}\|$$

$$\leqslant C\, h \; ,$$

with C independent of n. (Q.E.D.)

<u>Lemma 5</u> If k_1 and k_2 have the $\mathcal{H}$-property, then $k_1 k_2 = k_3$ also has the $\mathcal{H}$-property. If we denote by $K_{1h}, K_{2h}, K_{1h} \circ K_{2h}$ the associated family with k_1, k_2, k_3 respectively, then we have

$$\|\Lambda_h (K_{1h} K_{2h} - K_{1h} \circ K_{2h})\| \leqslant O(h) \; , \tag{2.22}$$

$$\|(K_{1h} K_{2h} - K_{1h} \circ K_{2h}) \Lambda_h\| \leqslant O(h) \; . \tag{2.23}$$

<u>Proof</u> We can assume as in the proof of Lemma 4 that k_1 and k_2 are some finite sums of $\mathcal{H}$-type, because of Lemma 1.

We put

$$k_1 = \sum_{\alpha} a_\alpha(x) k_\alpha^{(1)}(\xi), \qquad k_2 = \sum_{\beta} b_\beta(x) k_\beta^{(2)}(\xi) \quad ,$$

and their corresponding operators

$$K_{1h} = \sum_{\alpha} a_\alpha(x) K_{h\alpha}^{(1)} , \qquad K_{2h} = \sum_{\beta} b_\beta(x) K_{h\beta}^{(2)} .$$

Then we get

$$k_3 = \sum_{\alpha,\beta} a_\alpha(x) b_\beta(x) k_\alpha^{(1)}(\xi) k_\beta^{(2)}(\xi)$$

and

$$K_{1h} \circ K_{2h} = \sum_{\alpha,\beta} a_\alpha(x) b_\beta(x) K_{h\alpha}^{(1)} K_{h\beta}^{(2)} \quad ,$$

$$K_{1h} K_{2h} = \sum_{\alpha,\beta} a_\alpha(x) K_{h\alpha}^{(1)} b_\beta(x) K_{h\beta}^{(2)} .$$

Therefore we have

$$\begin{aligned} \Lambda_h (K_{1h} K_{2h} - K_{1h} \circ K_{2h}) &= \sum_{\alpha,\beta} \Lambda_h a_\alpha(x) \big(b_\beta(x) K_{h\alpha}^{(1)} - K_{h\alpha}^{(1)} b_\beta(x) \big) K_{h\beta}^{(2)} \\ &= \sum_{\alpha,\beta} (\Lambda_h a_\alpha(x) - a_\alpha(x) \Lambda_h) (b_\beta(x) K_{h\alpha} - K_{h\alpha} b_\beta(x)) K_{h\beta}^{(2)} \\ &\quad + \sum_{\alpha,\beta} a_\alpha(x) \Lambda_h \big(b_\beta(x) K_{h\alpha}^{(1)} - K_{h\alpha}^{(1)} b_\beta(x) \big) K_{h\beta}^{(2)} \quad . \end{aligned}$$

By Lemma 2 and Lemma 3 we know that

$$\| \Lambda_h a_\alpha(x) - a_\alpha(x) \Lambda_h \| \leqslant C_1 h \quad ,$$

$$\| \Lambda_h (b_\beta(x) K_{h\alpha} - K_{h\alpha} b_\beta(x)) \| \leqslant C_2 h \quad ,$$

with C_2 determined only by the Lipschitz constant of $|\xi| k_i(x,\xi)$ and the C^1-norm of $k_i(x,\xi)$ in x $(i=1,2)$. Hence we get easily the desired estimate (2.22). (2.23) can be proved in the same way. (Q.E.D.)

Lemma 6 If k has the $\mathcal{H}$-property, then k^*(complex conjugate) has also the $\mathcal{H}$-property. If we denote by K_h and $K_h^\#$ the associated family with k and k^* respectively, then we have

$$(2.24) \qquad \|\Lambda_h(K_h^* - K_h)\| \leqslant O(h) ,$$

$$(2.25) \qquad \|(K_h^* - K_h)\Lambda_h\| \leqslant O(h) .$$

Proof As in the proof of Lemma 5, we put

$$k(x,\xi) = \sum_{\alpha}^{finite} a_\alpha(x) k_\alpha(\xi) ,$$

$$K_h = \sum_\alpha a_\alpha(x) K_{h\alpha} .$$

Then we get

$$k^*(x,\xi) = \sum_\alpha a_\alpha^*(x) \overline{k}_\alpha(\xi) ,$$

and
$$K_h^\# = \sum_\alpha a_\alpha^*(x) \overline{K_{h\alpha}} ,$$

$$K_h^* = \sum_\alpha \overline{K_{h\alpha}} a_\alpha^*(x) .$$

Therefore the desired estimates (2.24) and (2.25) result directly from Lemma 3. (Q.E.D.). Let us return to (2.19). By using Lemmas 4, 5, 6 and the fact that $Q_h^{\#}\circ H_h = H_h \circ Q_h$ which results from our definition of H_h, we get (in the following the notation $\equiv$ means the equality in mod $\{$ the bounded operators of order $O(h)\}$,

$$\Lambda_h Q_h^* H_h E_h \equiv \Lambda_h Q_h^{\#} H_h E_h \equiv Q_h^{\#} H_h \Lambda_h E_h \equiv Q_h^{\#}\circ H_h \Lambda_h E_h$$

$$\equiv H_h \circ Q_h \Lambda_h E_h \equiv H_h Q_h \Lambda_h E_h \equiv H_h Q_h E_h \Lambda_h$$

$$\equiv H_h E_h Q_h \Lambda_h \equiv E_h H_h Q_h \Lambda_h \equiv E_h^* H_h Q_h \Lambda_h \ .$$

Therefore we get $\|P_h^{(1)}\| \leqslant O(h)$. Furthermore by Lemmas 4,5 and 6,

$$P_h^{(2)} \equiv \Lambda_h Q_h^{\#} H_h Q_h \Lambda_h \equiv Q_h^{\#} \Lambda_h H_h Q_h \Lambda_h \equiv Q_h^{\#} H_h \Lambda_h Q_h \Lambda_h$$

$$\equiv Q_h^{\#}\circ H_h \Lambda_h Q_h \Lambda_h \equiv Q_h^{\#}\circ H_h Q_h \Lambda_h^2 \equiv Q_h^{\#}\circ H_h \circ Q_h \Lambda_h^2 \ .$$

And we have also

$$P_h^{(0)} \equiv H_h (I - E_h^* E_h) \ ,$$

so that

$$P_h \equiv H_h (I - E_h^* E_h) - Q_h^{\#}\circ H_h \circ Q_h \Lambda_h^2 \ .$$

The Fourier transform of $I - E_h^* E_h$ is expressed by

$$I \cdot \left(1 - \left(\frac{\sum \cos h\xi_j}{n}\right)^2\right) ,$$

and

$$1-\left(\frac{\sum \cosh\xi_j}{n}\right)^2$$

$$=1-\frac{\sum \cos^2 h\xi_j}{n}+\frac{\sum_{j>k}(\cosh\xi_j-\cosh\xi_k)^2}{n^2}$$

$$=\frac{1}{n}|\sinh\xi|^2+\frac{1}{n^2}\sum_{j>k}(\cosh\xi_j-\cosh\xi_k)^2 .$$

Hence we can write using a notation $T_{jk}=T_{x_j}+T_{x_j}^{-1}-T_{x_k}-T_{x_k}^{-1}$,

$$P_h\equiv\frac{1}{n^2}\sum H_hT_{jk}^2+\left(\frac{1}{n}H_h-\lambda^2 Q_h^{\#}\circ H_h\circ Q_h\right)\Lambda_h^2 .$$

It can easily be shown that $\|H_hT_{jk}-T_{jk}H_h\|\leqslant O(h)$ and $T_{jk}^*=T_{jk}$, so that

$$P_h\equiv\frac{1}{n^2}\sum_{j>k}T_{jk}^*H_hT_{jk}+\left(\frac{1}{n}H_h-\lambda^2 Q_h^{\#}\circ H_h\circ Q_h\right)\Lambda_h^2 .$$

And we can write

$$\left(\frac{1}{n}H_h-\lambda^2 Q_h^{\#}\circ H_h\circ Q_h\right)\Lambda_h^2=N_h^{\#}\circ\left(\frac{1}{n}-\lambda^2\mathcal{D}_h^{\#}\circ\mathcal{D}_h\right)\circ N_h\Lambda_h^2 ,$$

where N_h and $\mathcal{D}_h$ are the pseudo difference schemes corresponding to $N(x,\xi)$ and $D(x,\xi)$ respectively. Now we put

$$p_1(x,\xi)=N_h^{\#}\circ\left(\frac{1}{n}-\lambda^2\mathcal{D}_h^{\#}\circ\mathcal{D}_h\right)\circ N_h .$$

We know that under the assumption of Theorem 1 $p_1(x,\xi)$ is positivedefinite. Then the associated family P_{1h} satisfies the following inequality for every $u \in L_x^2$,

$$(2.26) \qquad Re(P_{1h}\Lambda_h^2 u,\ u) \geqslant -O(h)\|u\|^2 .$$

In fact we can find a non singular matrix function $r(x,\xi)$ having the $\mathcal{H}$-property such that

$$p_1(x,\xi) = r^*(X,\xi)r(x,\xi) .$$

We denote a family corresponding to $r(x,\xi)$ by R_h. Then using Lemma 4 and 6

$$P_{1h}\Lambda_h^2 \equiv R_h^{\#} R_h\Lambda_h^2 \equiv R_h^{\#}\Lambda_h R_h\Lambda_h \equiv \Lambda_h R_h^{\#} R_h \Lambda_h$$

$$\equiv \Lambda_h R_h^* R_h \Lambda_h .$$

Hence we get

$$Re(P_{1h}u,\ u) = (R_h\Lambda_h u,\ R_h\Lambda_h u) + O(h)\|u\|^2$$

$$\geqslant -O(h)\|u\|^2 .$$

Thus we have

$$Re(P_h v, v) = Re\,\frac{1}{n^2}\sum_{j>k}(H_h T_{jk}v,\ T_{jk}v) + Re(P_{1h}\Lambda_h^2\, v, v)$$

$$\geqslant -O(h)\|v\|^2$$

$$\geqslant -O(h)Re(H_h v, v) ,$$

which is just the desired inequality (2.16). Thus the proof of Theorem 1 is complete.

Remark 2 In the above we have proved the inequality (2.26) in the case where $p_1(x,\xi)$ is positive definite. Then the proof was done very simply. Recently Vaillancourt[9] has given the following theorem :

Theorem 2 Suppose that $p_1(x,\xi)$ has the $\mathcal{H}$-property and is nonnegative. Then

$$Re(P_{1h}\Lambda_h^2 u, u) \geqslant - Kh\|u\|^2, \qquad u \in L_x^2 .$$

Therefore our Theorem 1 holds under the assumption of $\lambda \leqslant \frac{1}{\sqrt{n}\mu_0}$.

Remark 3 This method works as well in proving the stability of another schemes. For example, the modified Lax-Wendroff's scheme with accuracy 2 which was proposed recently by Richtmyer [12], can also be proved to be stable under the same assumption as in (1.3). This scheme can be written as

$$S_h u = \left[I + i\lambda Q_h\Lambda_h \left(E_h + i \frac{\lambda}{2} Q_h\Lambda_h\right)\right]u.$$

The essential feature of these schemes is that they are both polynominals of the same poerator Q_h that comes from the original system (1.1).

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Part III

Difference methods for the piston problem

§1. Introduction

We consider the piston problem [1] in hydrodynamics which arise from some models of gun-tunnels [3], free piston shock tubes [2], etc.. This problem has been difficult to be solved analytically, while for the design of the above apparatuses several approximation techniques are used and they depend mainly on characteristics and shock conditions, so that they seem to be very inconvenient. Then we have arrived at the necessity of direct algorithm solving the equation of hydrodynamics by difference methods.

In view of the studies of difference schemes themselves many authors have attacked the Cauchy problem, while for mixed initial-boundary value problems, we know only a few results and we are not in the position having any practically effective methods. Here we aimed to discover an appropriate method for mixed problems.

As a simple one-dimensional model of a gun-tunnel we take the following system.

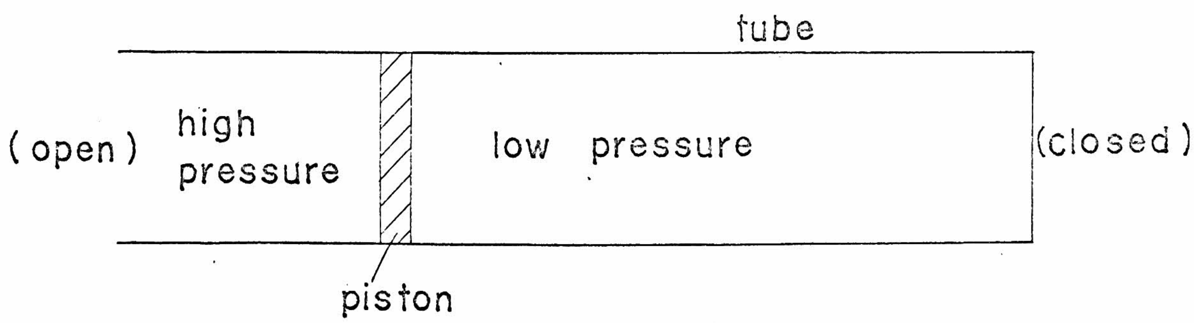


Fig.1, The model of the gun-tunnel

Here the cross area of the tube is assumed to be constant, initially the piston is positioned at some point and the states of gas in both chambers rest uniformly, and the pressure of the gas in the left chamber is higher than that in the right chamber. As soon as the piston is set free it begins to move by the difference of the forces acting on both sides of the piston. Our aim is to introduce an algorithm to solve it numerically.

The algorithm is mainly depending on Godunov's idea and is constructed to calculate also in the neighbouhood of boundary points.

§2. Differential equations and boundary conditions

Mainly we shall use the usual equation of hydrodynamics in the Eulerian form and the polytropic relation to express the motion and state of gas in each chamber:

$$
\begin{aligned}
&\rho_t + (\rho u)_x = 0 \\
&(\rho u)_t + (p+\rho u^2)_x = 0 \\
&\left\{\rho\left(e+\tfrac{1}{2}u^2\right)\right\}_t + \left\{\rho u\left(e+\tfrac{1}{2}u^2+\tfrac{p}{\rho}\right)\right\}_x = 0 \\
&p = (\gamma-1)\rho e
\end{aligned}
\tag{2.1}
$$

where ρ is the density, u is the velosity, p is the pressure, e is the internal energy per unit mass and γ is the adiabatic exponent.

The equation of the piston motion is as following :

$$\frac{d^2\xi(t)}{dt^2} = p(t,\xi(t)-0) - p(t,\xi(t)+0), \quad \xi(0) = 0, \tag{2.2}$$

where $x = \xi(t)$ is the piston path, and we can assume by appropriately normalizing so that the piston mass is unit.

The boundary condition being satisfied at the position of the piston is

$$u(t,\xi(t)-0) = u(t,\xi(t)+0) = \frac{d\xi(t)}{dt}.$$

Furthermore we have the following boundary condition at the right end wall $(x=x_w)$:

$$u(t,x_w) = 0.$$

As initial conditions we suppose that the initial static states have uniformly constant density and pressure in respective chambers.

§3. The modified Godunov's scheme

For solving the above problem we must make up algorithms in the neighborhood of the piston, in that of the wall and through the inner region respectively.

First we consider the scheme through the inner region.

One wants such schemes as reproduce especially shock waves which are discontinuous solutions. Such schemes, as far as we know, are divided into two classes. One is of the method using "artificial viscosity" and the other depends on "the decay of the discontinuity". An example of the former is the Lax--Wendroff's scheme, and that of the latter is the Godunov's scheme . Both are known to be excellent methods, but according to our experiments for the Riemann problem it seems that the modified Godunov's scheme is a little better than the L-W scheme, and we used mainly the modified Godunov's scheme.

Now we will show briefly the Godunov's scheme and the modified one. On a time level we consider the grid function as the step function having discontinuities at the half-integer points. First we shall solve the decay of the discontinuities and secondly calculate the mean values around each grid point on the next time step. So formed grid functions are used to take the next step, and so on.

In order to proceed to such an algorithm we had better use the following integral formula instead of the original differential equations (2.1) :

$$\oint \rho\, dx - \rho u\, dt = 0 \ ,$$

$$(3.1) \qquad \oint \rho u\, dx - (p+\rho u^2)\, dt = 0 \ ,$$

$$\oint \rho \left(e+ \frac{u^2}{2}\right) dx - \rho u\left(e+ \frac{u^2}{2} + \frac{p}{\rho}\right) dt = 0 \ .$$

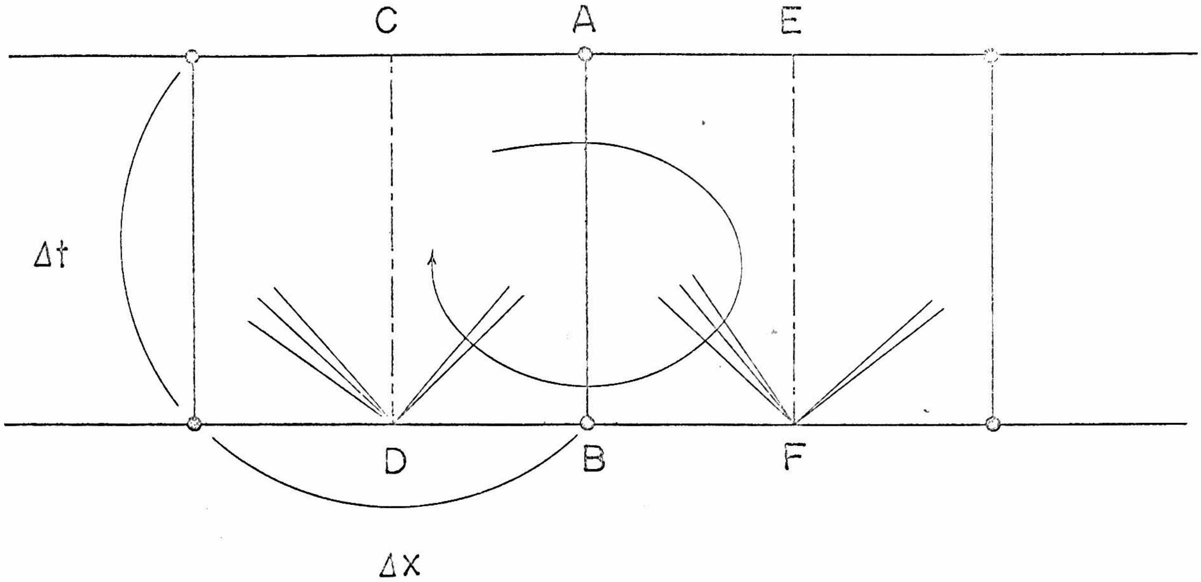


Fig.2,

In the Fig-2 we want to know the unknown value at the grid point A. Then we take the integral contour CEFD. For example, we have for the first equation of (3.1),

$$(3.2) \qquad \Delta x \widetilde{\rho}_A = \Delta x \widetilde{\rho}_B - \Delta t \left\{ (\overline{\rho u})_{FE} - (\overline{\rho u})_{CD} \right\} ,$$

so that we can know the mean value $\widetilde{\rho}_A$ around the grid point A if we know ρu's on the lines EF and CD, which are obtained by solving the decay of discontinuities at F, D respectively. For solving the decay of discontinuities we must go through an iteration process. In order to avoid it we shall use below the modified Godunov's scheme, in which the waves are supposed to propagate to right and left along characteristics. This assumption is valid only for the case in which the quantities of jump are small. Thus we have the following algorithm :

$$\rho^j = \rho_j - \lambda (R_{j+\frac{1}{2}} U_{j+\frac{1}{2}} - R_{j-\frac{1}{2}} U_{j-\frac{1}{2}}) ,$$

$$\rho^j u^j = \rho_j u_j - \lambda (P_{j+\frac{1}{2}} + R_{j+\frac{1}{2}} U^2_{j+\frac{1}{2}} - P_{j-\frac{1}{2}} - R_{j-\frac{1}{2}} U^2_{j-\frac{1}{2}}) .$$

$$\rho^{j}(e^{j}+ \tfrac{1}{2}(u^{j})^{2}) = \rho_{j}(e_{j}+\tfrac{1}{2}\, u_{j}^{2})$$

(3.3)

$$- \lambda \left\{ R_{j+\frac{1}{2}}\, U_{j+\frac{1}{2}}\, (E_{j+\frac{1}{2}} + \tfrac{1}{2}\, U^{2}_{j+\frac{1}{2}} + \frac{P_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}}) - R_{j-\frac{1}{2}}\, U_{j-\frac{1}{2}}(E_{j-\frac{1}{2}} + \tfrac{1}{2}\, U^{2}_{j-\frac{1}{2}} + \frac{P_{j+\frac{1}{2}}}{R_{j+\frac{1}{2}}}) \right\},$$

$$p^{j} = (\gamma - 1)\rho^{j} e^{j}, \quad \lambda = \frac{\Delta x}{\Delta t},$$

where the lower suffix inidicates the x-position on a time level $t=t_0$, and the upper indicates that on $t=t_0+\Delta t$.

The auxiliary values are calculated as follows :

$$a_{j+\frac{1}{2}} = \sqrt{\gamma\, \frac{p_j+p_{j+1}}{2}\ \frac{\rho_j+\rho_{j+1}}{2}},$$

$$P_{j+\frac{1}{2}} = \frac{p_{j+1}+p_j}{2} - a_{j+\frac{1}{2}}\, \frac{u_{j+1}-u_j}{2},$$

$$u_{j+\frac{1}{2}} = \frac{u_{j+1}+u_j}{2} - \frac{p_{j+1}-p_j}{2a_{j+\frac{1}{2}}},$$

$$\rho^{L}_{j+\frac{1}{2}} = \frac{(\gamma+1)p_{j+\frac{1}{2}} + (\gamma-1)p_j}{(\gamma-1)p_{j+\frac{1}{2}}+(\gamma+1)p_j}\, \rho_j,$$

$$\rho^{R}_{j+\frac{1}{2}} = \frac{(\gamma+1)p_{j+\frac{1}{2}}\ (\gamma-1)p_{j+1}}{(\gamma-1)p_{j+\frac{1}{2}}+(\gamma+1)p_{j+1}}\, \rho_{j+1},$$

$$S^{L}_{j+\frac{1}{2}} = u_j - \frac{a_{j+\frac{1}{2}}}{\rho_j}, \qquad S^{R}_{j+\frac{1}{2}} = u_{j+1} + \frac{a_{j+\frac{1}{2}}}{\rho_{j+1}}.$$

Consequently we have the following four cases for large capitals U,P,R and E :

i) If $S^L_{j+\frac{1}{2}} > 0$, $S^R_{j+\frac{1}{2}} > 0$, then

$$U_{j+\frac{1}{2}} = u_j , \quad P_{j+\frac{1}{2}} = p_j , \quad R_{j+\frac{1}{2}} = \rho_j ,$$

ii) If $S^L_{j+\frac{1}{2}} < 0$, $S^R_{j+\frac{1}{2}} < 0$, then

$$U_{j+\frac{1}{2}} = u_{j+1}, \quad P_{j+\frac{1}{2}} = p_{j+1}, \quad R_{j+\frac{1}{2}} = \rho_{j+1} ,$$

iii) If $S^L_{j+\frac{1}{2}} < 0$, $S^R_{j+\frac{1}{2}} > 0$, $u_{j+\frac{1}{2}} > 0$, then

$$U_{j+\frac{1}{2}} = u_{j+\frac{1}{2}}, \quad P_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} , \quad R^L_{j+\frac{1}{2}} = \rho^L_{j+\frac{1}{2}} ,$$

iv) If $S^L_{j+\frac{1}{2}} < 0$, $S^R_{j+\frac{1}{2}} > 0$, $u_{j+\frac{1}{2}} < 0$, then

$$U_{j+\frac{1}{2}} = u_{j+\frac{1}{2}} , \quad P_{j+\frac{1}{2}} = p_{j+\frac{1}{2}} , \quad R_{j+\frac{1}{2}} = \rho^R_{j+\frac{1}{2}} ,$$

and
$$E_{j+\frac{1}{2}} = \frac{P_{j+\frac{1}{2}}}{(\gamma-1)R_{j+\frac{1}{2}}} .$$

By the linear stability analysis it is know that this scheme is stable under the C.F.L. condition.

§4. The algorithm in the neighbourhood of the piston

Next we consider the neighbourhood of the piston. Since the piston moves across the net, the algorithm becomes complicated. In order to compare quantitatively the various methods, we had the numerical experiments of the piston problem for the linear wave equation in which the piston was to be moved with constant acceleration. And we have arrived at the method having the comparatively small errors (see Appendix-1). This method can be also applied to the nonlinear equation of the fluid dynamics. In our experiment this method gave good results (see Appendix-II). Here we are to interpret the method when the piston moves to the right, the piston path being drawed on the net in the following two ways for sufficiently small Δt. (Fig.3)

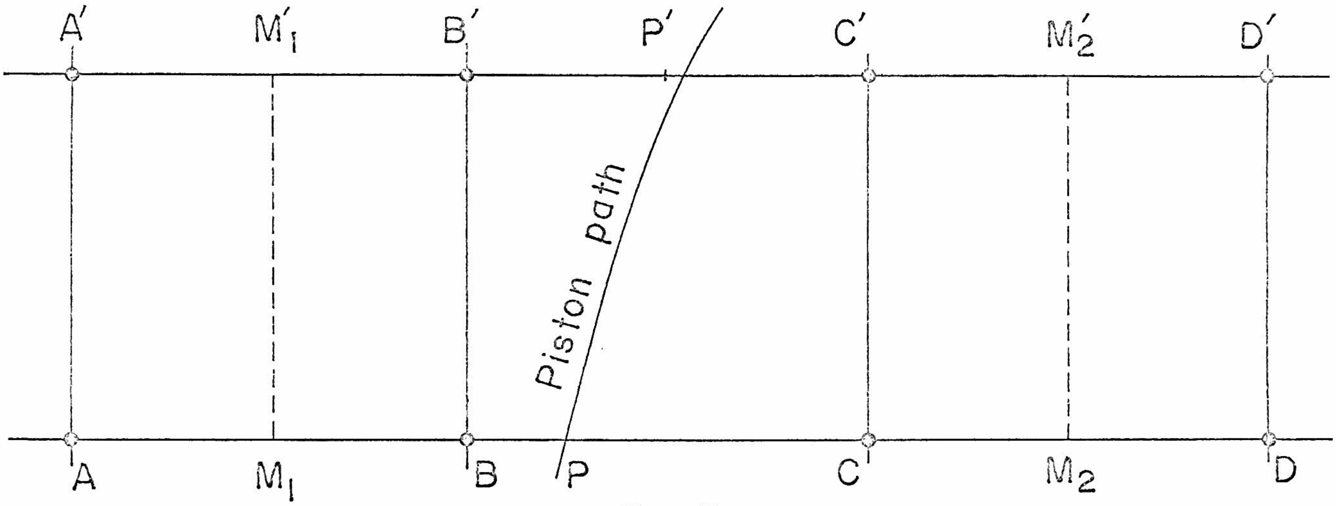


Fig.3-a,

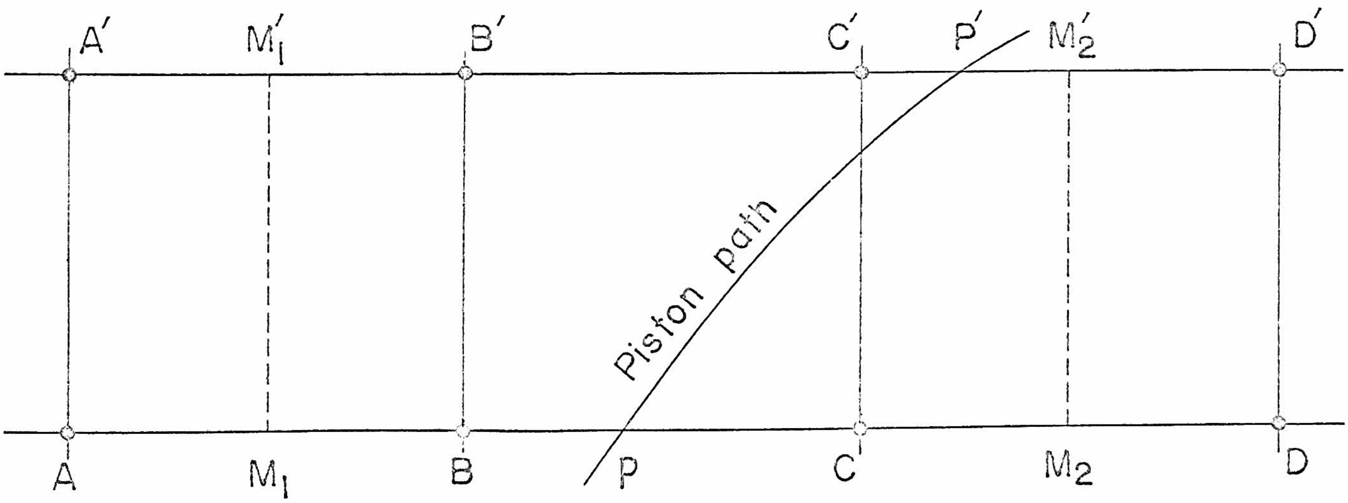


Fig.3-b,

In the Fig. 3-a,b we suppose that the values of u, ρ, p at the mesh points A,B,C,D etc. and that at the point P of the piston are known. Then we desire those values at the points A',B',C',D' and P' after Δt. For this we again use the integral formulas (3.1). If we integrate along the contour $P'M'_2 M_2 P$ in the Fig.3-a, we can have the integrated values (consequently the mean values) on the $P'M'_2$ using the integrated values on the M'_2M_2, M_2P and PP'. Therefore we shall first calculate the mean values on the M'_2M_2 and PP'. Supposing that the mean values are kept constant on the PM_2 and that the piston runs along PP' with a known positive constant speed we can decide the state in front of the piston by using the Rankine--Hugoniot's relation across the shock wave generated at the point P ;

we find first the shock speed U

$$U = \frac{1}{2}\frac{\bar{u}_p - \bar{u}}{1-\mu^2} + \sqrt{\bar{c}^2 + \frac{1}{4}\left(\frac{\bar{u}_p - \bar{u}}{1-\mu^2}\right)^2} , \quad \mu^2 = \frac{\gamma-1}{\gamma+1} ,$$

where $\bar{u}_p$ is the piston speed (the suffix p and the bar mean here and just below the mean values on the PP' and "+"- in front of, "-" - at the back of) and the barred values $\bar{u}$ and $\bar{c}$ means the mean values on the PM_2 of the velocity and the sound velocity respectively (the bars mean here and below the mean values on the PM_2) and furthermore we find

$$\bar{p}_+ = \bar{p}\left\{(1+\mu^2)\frac{(U-\bar{u})^2}{\bar{c}^2} - \mu^2\right\},$$

(4.1)
$$\bar{\rho}_+ = \bar{\rho}\,\frac{p_p+\mu^2 p}{p+\mu\, p_p}.$$

The desired values on the $M_2' M_2$ are calculated by the decay of the discontinuity at the point M_2 as we have done above in the interior region. Hereafter we have the mean values $\widetilde{\cdot}$ on the P'M' as follows :

$$\widetilde{\rho} = \alpha\bar{\rho} - \beta(\rho u)_{M_2M_2'},$$

(4.2)
$$\widetilde{\rho u} = \alpha\overline{\rho u} - \beta\left\{(p+\rho u^2)_{M_2M_2'} - p_p\right\},$$

$$\widetilde{\rho\left(e+\frac{u^2}{2}\right)} = \alpha\,\overline{\rho\left(e+\frac{u^2}{2}\right)} - \beta\left[\left(\rho u\left(e+\frac{u^2}{2}+\frac{p}{\rho}\right)\right)_{M_2M_2'} - (pu)_p\right],$$

where $\alpha = \widetilde{PM_2}/\overline{P'M_2'}$, $\beta=\Delta t/\overline{P'M_2'}$ and $(\hat{\ })_{M_2M_2'}$ indicates the mean values on the M_2M_2' . On the other hand under the same hypothesis of the piston motion the expansion wave propagates backward.

We can also decide the state at the back of the piston as follows :

$$\bar{p}_- = \bar{\bar{p}}\left[1 - \frac{\gamma-1}{2}\,\frac{u_p-\bar{\bar{u}}}{\bar{\bar{c}}}\right]^{\frac{2\gamma}{\gamma-1}},$$

(4.3)
$$\bar{\rho}_- = \bar{\bar{\rho}}\left[1 - \frac{\gamma-1}{2}\,\frac{u_p-\bar{\bar{u}}}{\bar{\bar{c}}}\right]^{\frac{2}{\gamma-1}},$$

where the doubly-barred values show the mean values on the M_1P. The desired mean values on the $M'_1 P'$ are calculated by using again the integral formulas (3.1) along the path $M_1'P'PM_1$. On the contrary, when the piston moves with a known negative constant speed, we have the shock wave to the left and the expansion wave to the right. Then we can calculate the mean values on the $M_1'P'$ and $P'M_2'$ in a similar way. While in the case of the Fig.3-b, we shall decide the states in the front of and at the back of the piston just in the same way as in the case of Fig.3-a, the values at the point D' are calculated by the interior formula. Thus we have the mean values in the neighbourhood of the piston, and furthermore we can calculate the desired values at the grid points by the interpolation between the corresponding mean values and the states at the piston.

So far we supposed that the piston-speed was to be known in advance, but in reality it is unknown beforehand and is the quantity to be desired. If we know the states at both sides of the piston, we can know the piston speed by integrating the equation of the piston motion. Approximately we have the follwing formula :

$$(4.4)\qquad \begin{aligned} u_p' &= u_p + \Delta t(\bar{p}_- - \bar{p}_+), \\ \bar{u}_p &= \frac{u_p' + u_p}{2}. \end{aligned}$$

Hence in order to know the states of both sides at the piston and the speed of the piston, we can not but use the iteration scheme among the formulus (4.1), (4.3) and (4.4). It is easily seen that this iteration scheme converges for small Δt.

§5. The algorithm in front of the wall

We suppose that the net is set in front of the wall as the following figure :

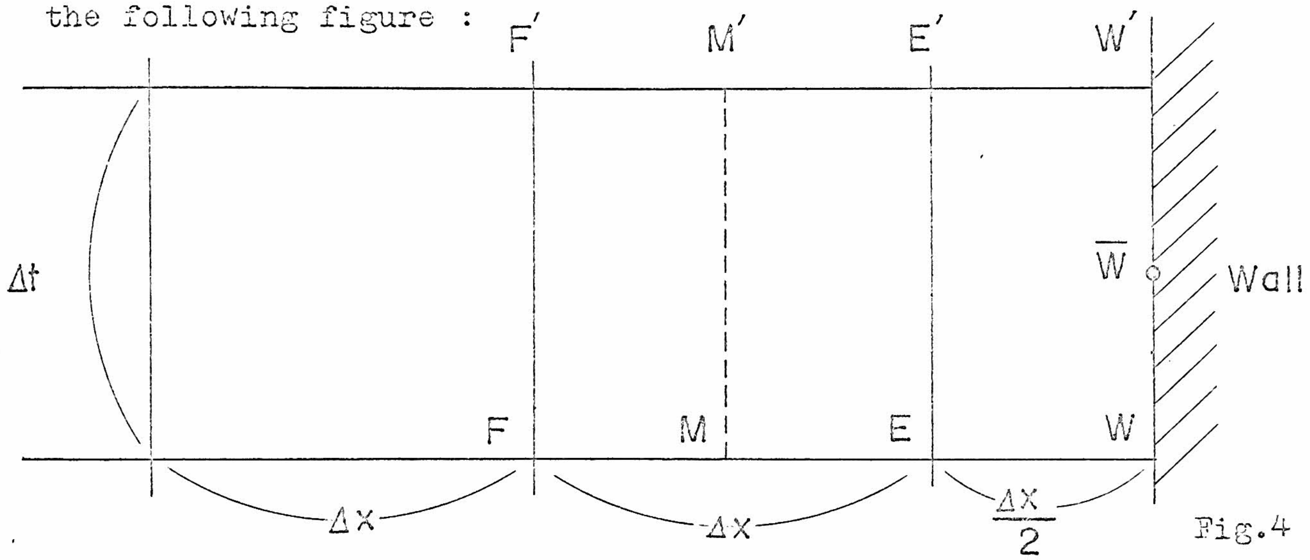


Fig.4

We shall construct the scheme which gives the relevant values at the mesh point E' and at the point $\overline{W}$ on the wall using the values at E and F etc. For this, along the same line as in the neighbourhood of the piston we consider the reflection of the shock wave at the point W on the wall and ask the state on the wall using the Rankine-Hugoniot relation, and furthermore apply the contour-integral (3.1) along M'W'WM so that we have the relevant values at the mesh point E'. Here, of course, it is supposed that we have a uniform state along MW and we calculate the decay of the discontinuity at the point M as in the interior region. Consequently this algorithm is put in order as follows :

$$c_E = \sqrt{\frac{\gamma p_E}{\rho_E}} \quad ,$$

$$u_S = - \frac{u_E}{2(1-\mu^2)} - \sqrt{c_E^2 + \left(\frac{u_E}{2(1-\mu^2)} \right)^2} \quad ,$$

$$p_{\bar{W}} = p_E \left\{ (1+\mu^2) \left(\frac{u_S}{c_E} \right)^2 - \mu^2 \right\} \quad ,$$

$$\rho_{\bar{W}} = \frac{\gamma p_{\bar{W}}}{c_E^2 \quad \frac{\gamma-1}{2} u_E (2u_S + u_E)} \quad .$$

§6. An experiment

In order to check our method we use the following data as an example : the length of the barrel between the initial position and the closed end is 3.5 m, the radius is 3.7 cm, the pressure in the barrel (the right chamber) is 2.8kg W/cm , that in the resevoir (the left chamber) is 66 kg W/cm , the initial sound velocity in both chambers is 331 m/sec and the piston weight W is 0.005 kg W. These data depend on that of the hypersonic gun tunnel at the Kyoto University constructed in 1962.[3] We carry out, as usual, to make dimensionless , as follows :

$$u \rightarrow \bar{u} = \frac{u}{c_o} \quad , \qquad c_o^2 = \frac{\gamma p_o}{\rho_o} \quad ,$$

$$p \rightarrow \bar{p} = \frac{p}{p_o} \quad , \qquad \rho \rightarrow \bar{\rho} = \gamma \frac{\rho}{\rho_o} \quad ,$$

$$x \rightarrow \bar{x} = \frac{g p_o A_o x}{W c_o^2} \quad , \quad t \rightarrow \bar{t} = \frac{g p_o A_o t}{W c_o} \quad ,$$

where W is the piston weight, A_0 is the cross area of the barrel and g is the gravitational acceleration.

The zero suffix of the other values means the initial values in the right chamber. Thus we have the equations of hydrodynamics (2.1) and the equation of the piston motion (2.2), where the bar is omitted. And the above initial data are reduced to the following dimensionless quantities ;

$$p_0 = 1 \ , \ \rho_0 = 1.4 \ , \ u_0 = 0 \quad \text{(in the right chamber)},$$

$$p_1 = 23.57, \rho_1 = 13.37 \ , \ u = 0 \quad \text{(in the left chamber)}$$

and the length of the barrel is 1.88.

In our experiments $\Delta x = 0.094$, $\lambda = 0.2$ which satisfies the C.F.L. condition [7] in our data and results. In Fig.5 we see how the pressure in the chambers vary as time passes. In the Fig.6 we see the piston path (the real line) and the aspect of the propagation of the shock waves. The pressure at the end wall varies as in Fig.7. This pattern agrees fairly with the experimental result . Futhermore the pressure histories in the front of the piston (the solid line) and at the back of it (the broken line) are shown in Fig.8. In order to check our result we shall compute the theoretical pressure and the one of our numerical experiment at the back of the forward shock wave while the piston runs with constant speed. The former was 3.94 (with no dimension) and the latter was 3.96.

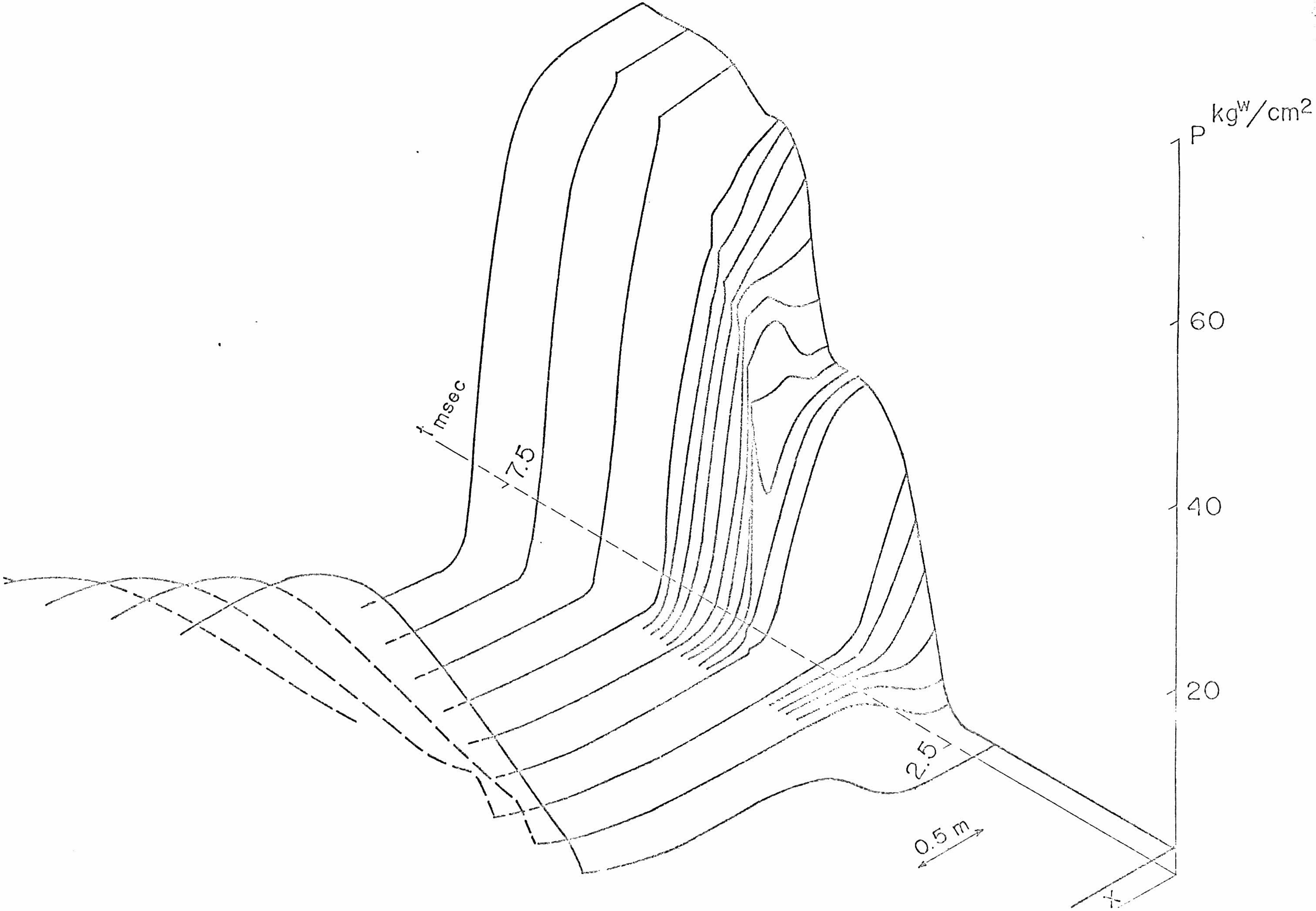
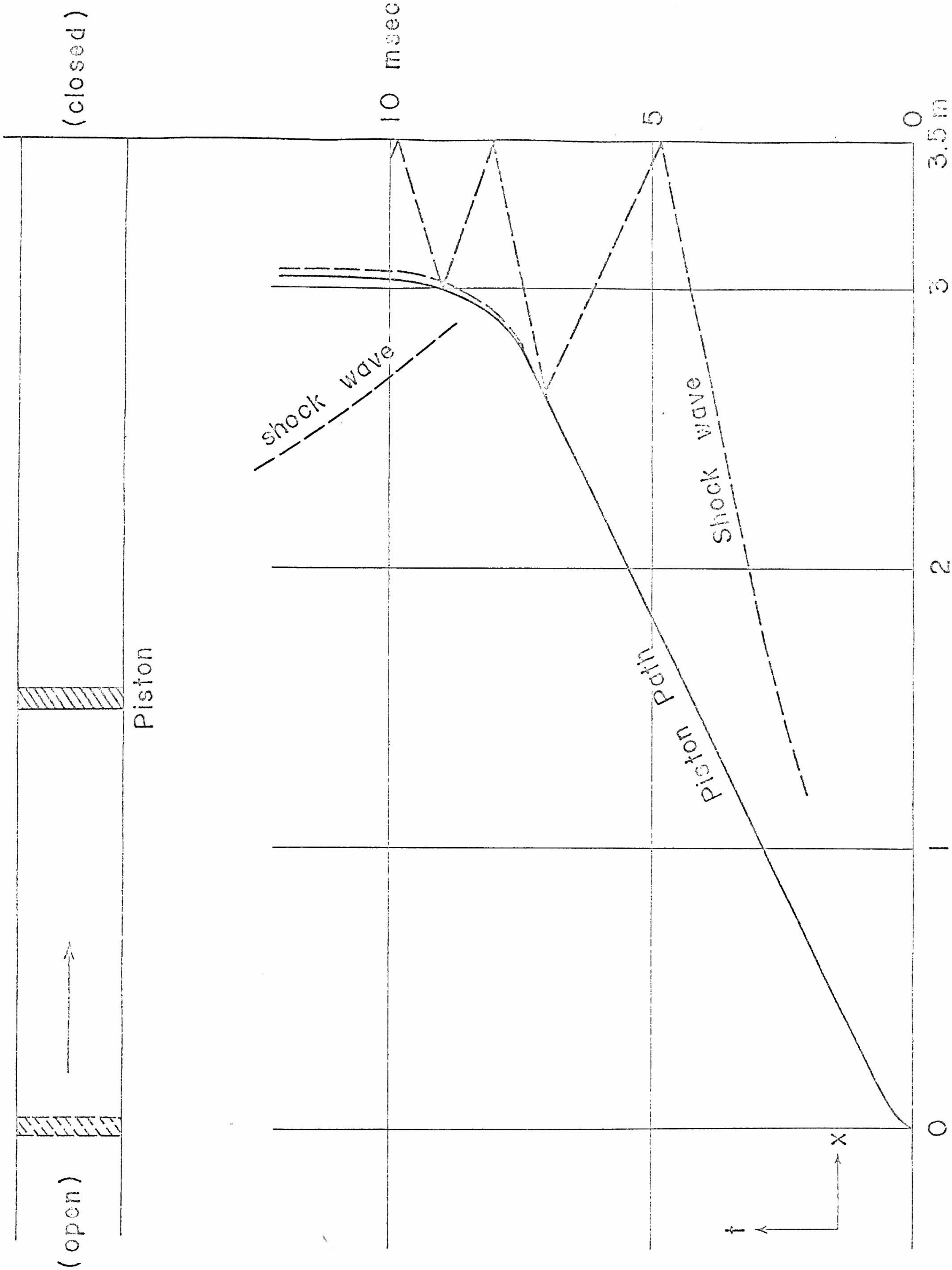


Fig.5, The pressure history

Fig.6, The piston path (the real line - by the Eulerian formula, the broken line - by the Lagrangian formula) and the propagation of the shock wave.



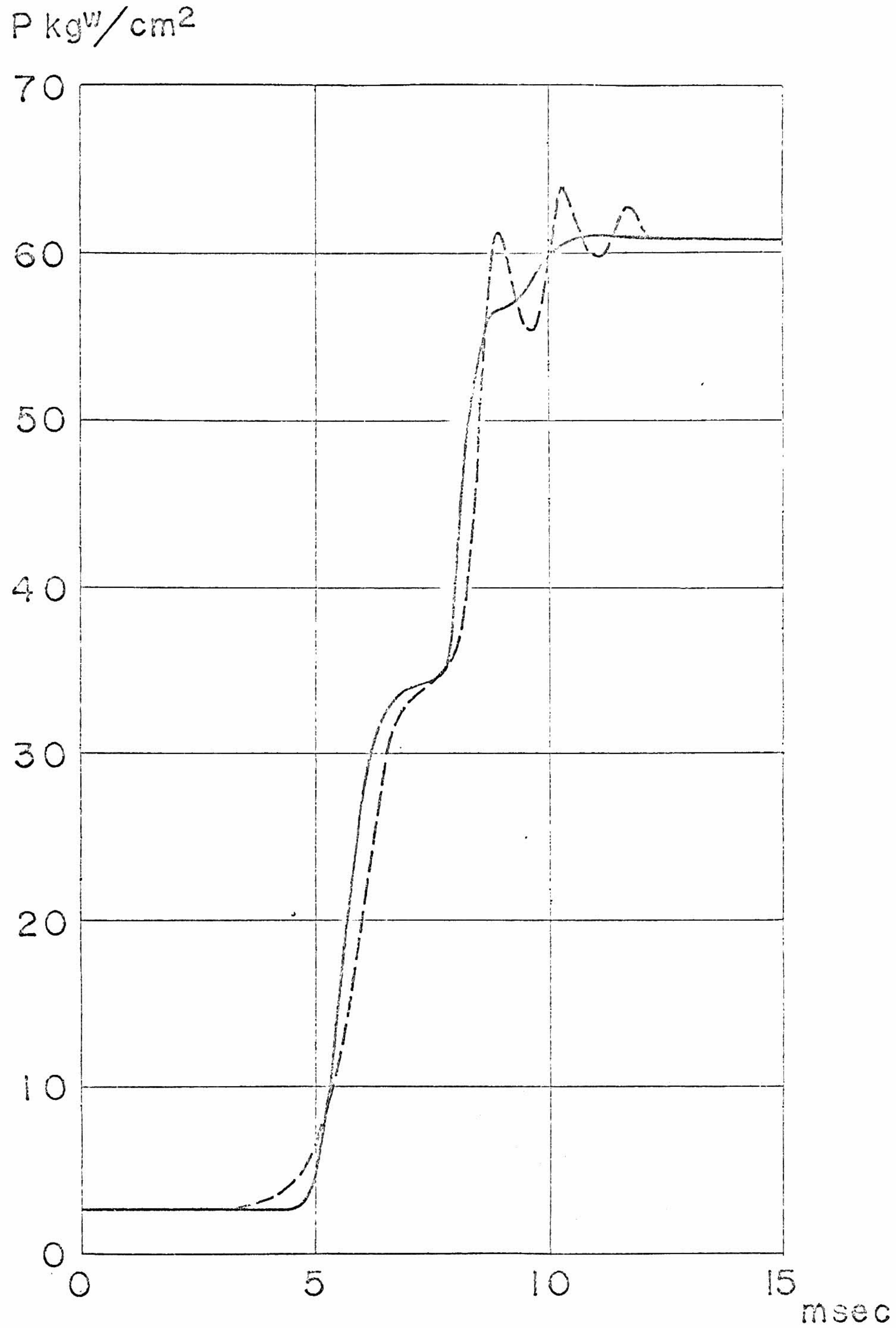


Fig.7

The pressure history at the wall (the real line - by the Eulerian formula, the broken line - by the Lagrangian formula)

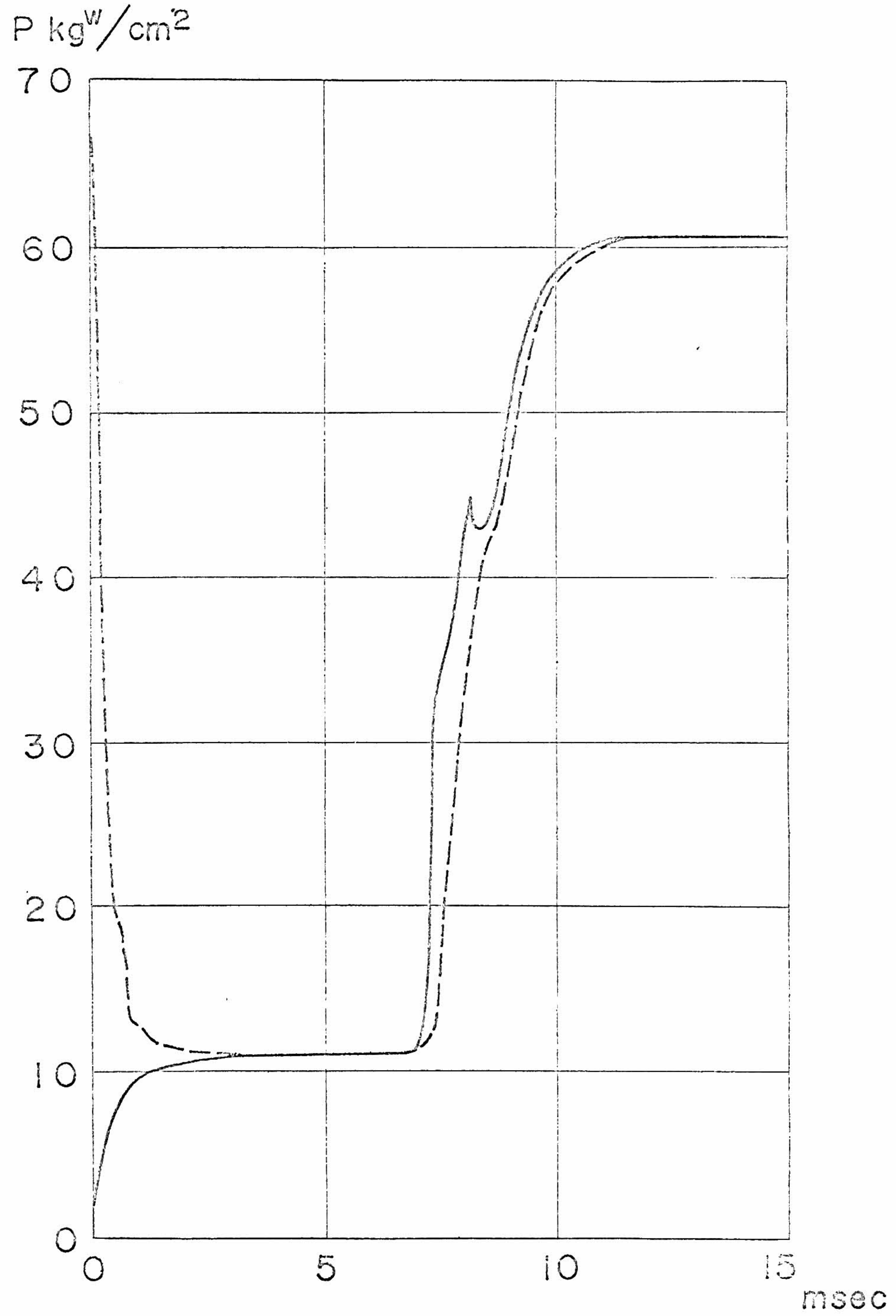


Fig.8

The pressure histories in the front of (the real line) and at the back of the piston (the broken line)

Appendix I

We shall consider the piston problem for the wave equation

$$(\text{I-1})_1 \quad \begin{cases} u_t = v_t \\ v_t = u_x \end{cases}, \qquad x > X(t), \quad t > 0 .$$

Initial conditions :

$$(\text{I-1})_2 \quad u(0,x) = 0 \quad , \quad v(0,x) = 1.0 \quad , \quad x > 0 .$$

Piston path :

$$(\text{I-1})_3 \quad X(t) = 0.1\, t^2 .$$

Boundary condition :

$$(\text{I-1}) \quad u\Big|_{x = X(t)} = 0.2t .$$

And we shall try to compare the various algorithms in the neighbourhood of the piston.

We shall approximate the differential equation by the following difference equation (the Godunov's scheme) :

$$(\text{I-2}) \quad \begin{aligned} u_j^{n+1} &= u_j^n + \frac{\lambda}{2}(v_{j+1}^n - v_{j-1}^n) + \frac{\lambda}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n) , \\ v_j^{n+1} &= v_j^n + \frac{\lambda}{2}(u_{j+1}^n - u_{j-1}^n) + \frac{\lambda}{2}(v_{j+1}^n - 2v_j^n + v_{j-1}^n) . \end{aligned}$$

Now when the net is fixed, the piston path runs across it in the following two ways :

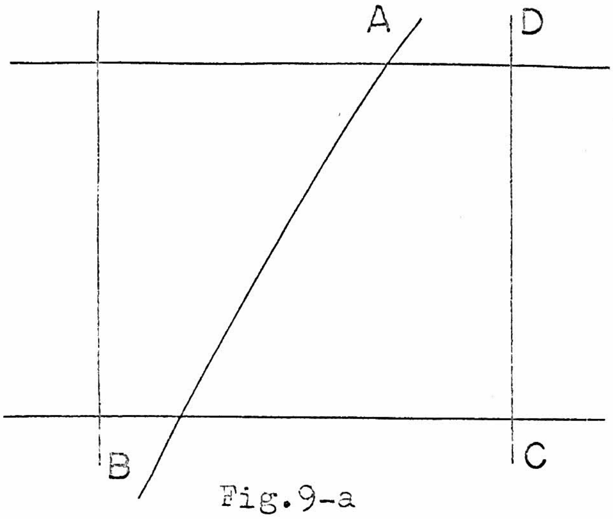


Fig.9-a

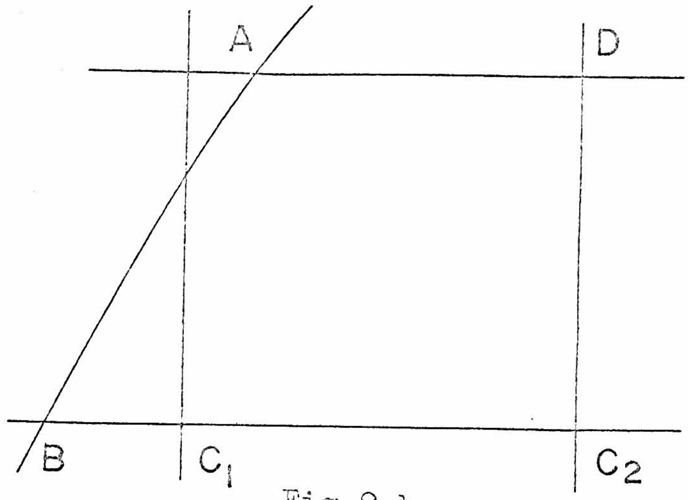


Fig.9-b

The problem is to determine the relevant values at the points A and D using those at the points B,C, etc.

(i) Method-1 First we shall introduce the most formal method.

In the case-a

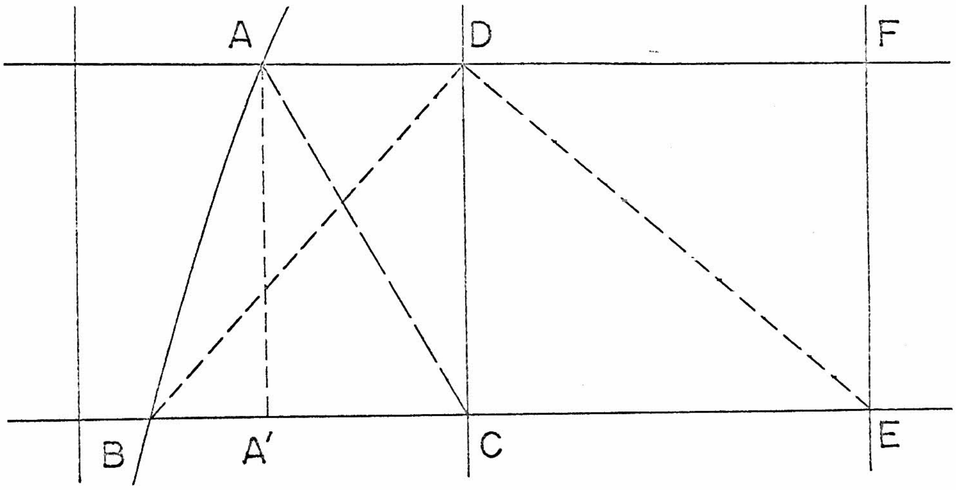


Fig.10-a

We shall put

$$BA' : A'C = \alpha : \beta \quad , \quad BC : CE = \alpha' : \beta'$$

$$(\alpha + \beta = \alpha' + \beta' = 1)$$

and

$$v_A = \alpha v_C + \beta v_B + \frac{\Delta t}{BC} (u_C - u_B) ,$$

$$u_D = \alpha' u_E + \beta' u_B + \frac{\Delta t}{BE} (v_E - v_B) ,$$

$$v_D = \alpha' v_E + \beta' v_B + \frac{\Delta t}{BE} (u_E - u_B) .$$

In the case-b

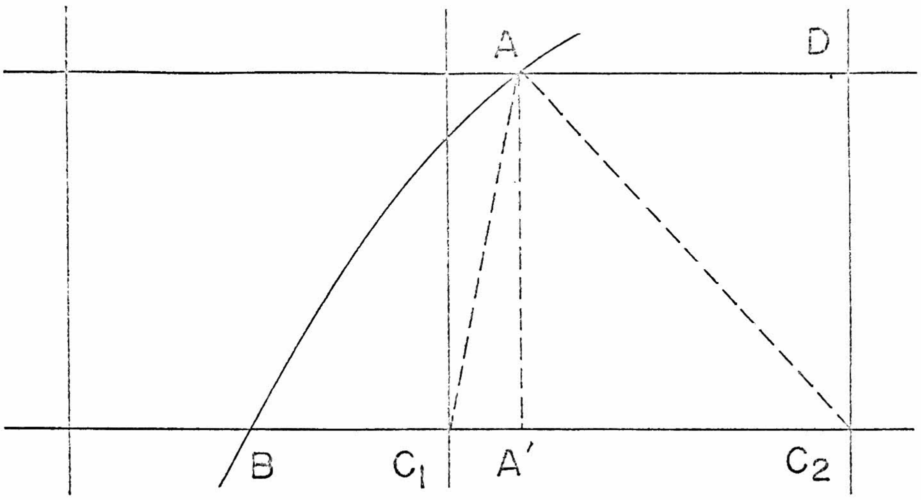


Fig.10-b

we shall put

$$C_1A' : A'C_2 = \alpha : \beta \quad , \qquad (\alpha+\beta=1)$$

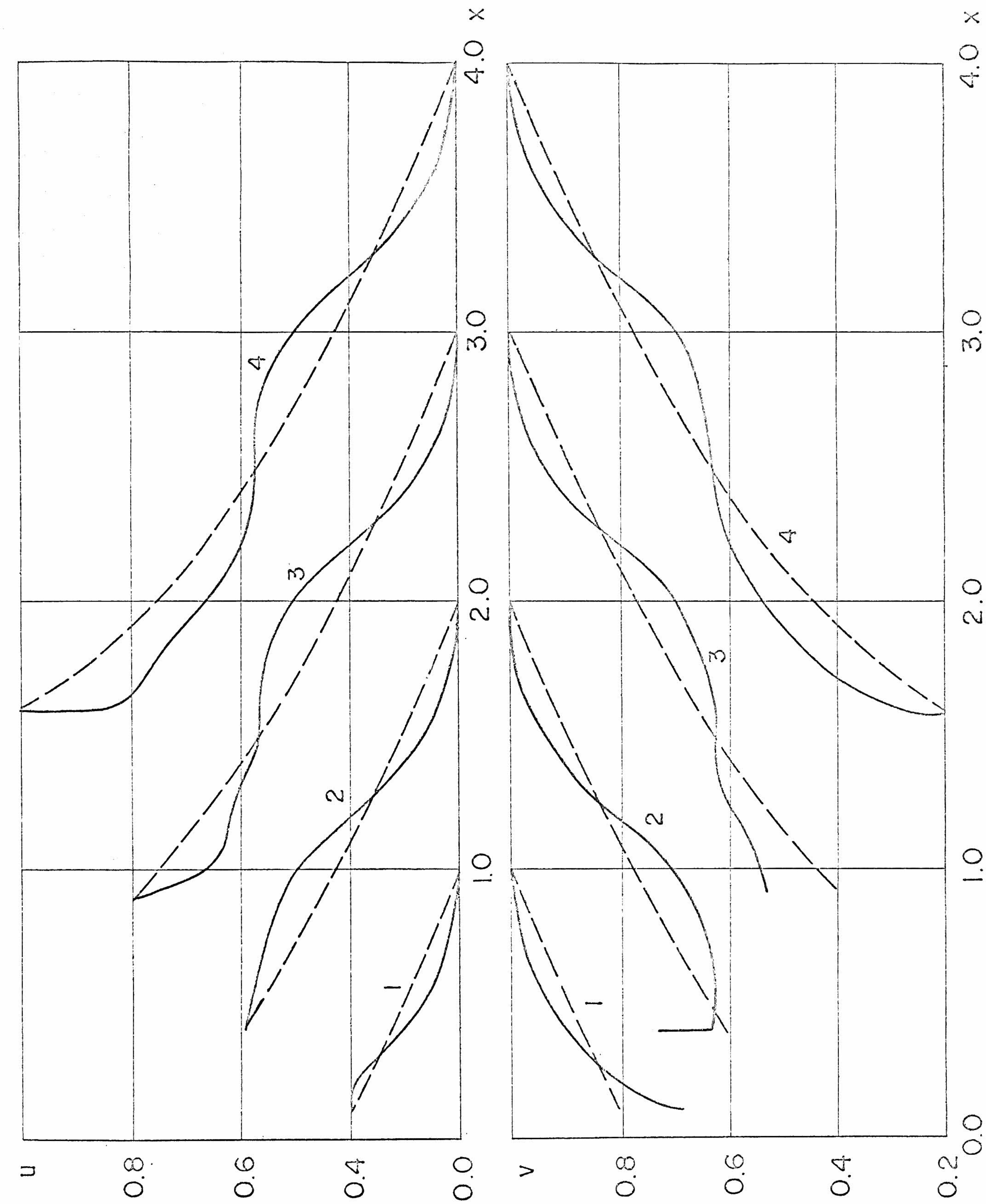
and

$$v_A = \alpha v_{C_2} + \beta v_{C_1} + \lambda(u_{C_2} - u_{C_1}), \quad (\lambda = \frac{\Delta t}{\Delta x}) \quad .$$

The relevant values at D are calculated by the formula (I-2). The experiment by this method is shown in Fig.11, where $\lambda=1.0$, $x=0.2$ and the parameter on the curve means the time. The broken line shows the exact solution of the problem (I-1). (These conventions are common among the following figures.)

(ii) Method-2. In the result by the Method-1, the values on the piston become far from the exact ones and its effects are propagated to the right. In order to determine the values better we shall use the characteristics, that is, use the property of the solution that $u+v$ remains constant along the line $\frac{dx}{dt} = -1$.

Fig.11, Method-1



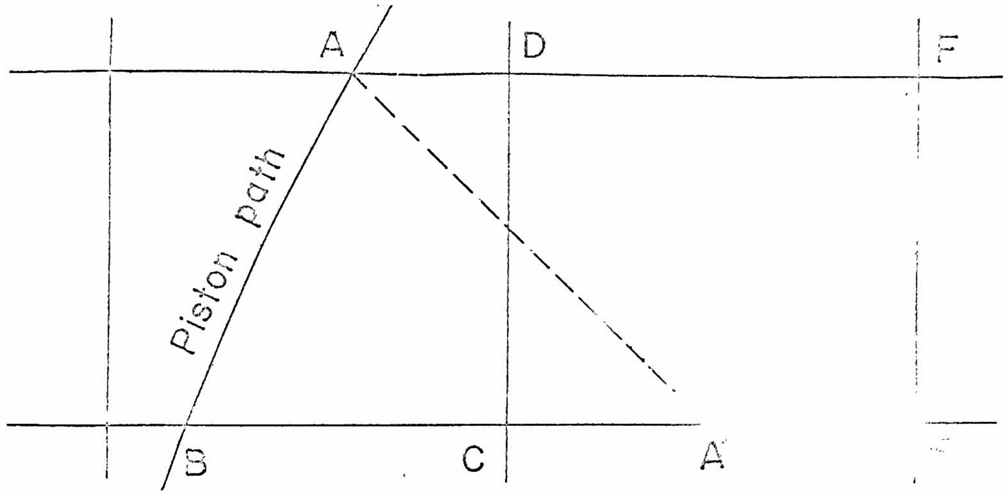


Fig.12

Draw the characteristic AA' ($\frac{dx}{dt} = -1$) through the point A and determine the position A' as in the Fig.12 and set

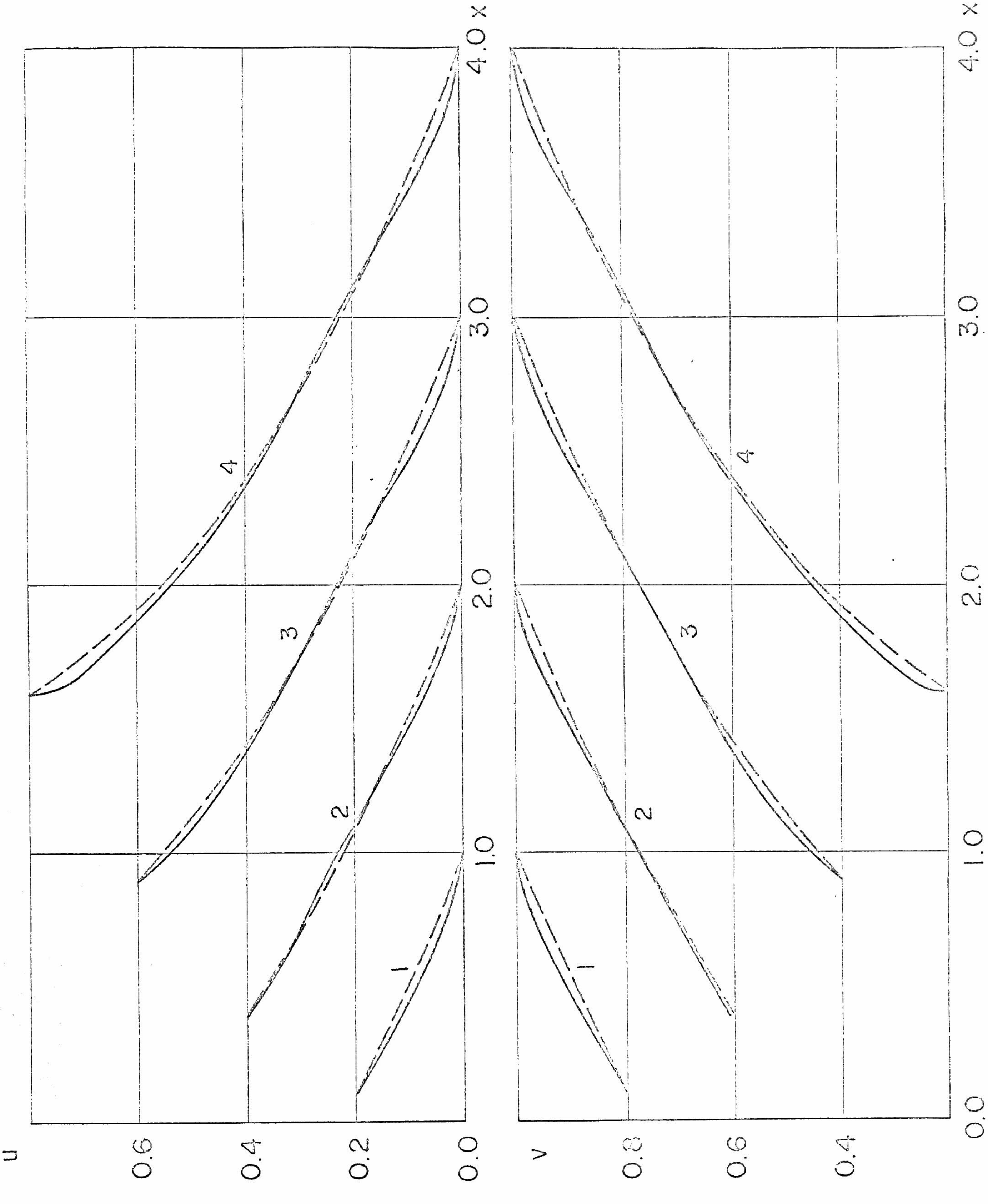
$$v_A + u_A = v_{A'} + u_{A'} \quad .$$

If we determine the values $v_{A'}$ and $u_{A'}$ by interpolation between C and E, we have the value v_A (u_A is given).

The method to determine the values at the point D is as Method-1. By this algorithm we have the experimental results in Fig-13. This result is certainly better than that of Method-1 and shows the **broad aspect of the exact solution** in spite of the coarse mesh width $\Delta x=0.2$. A defect is, as we see in the Fig-13, the notable difference of the values at A, D when D is close to A. This means a mistake in the way calculating the value at D.

(iii) Method-3 Another defect of Method II is that when it is applied to fluid dynamics, we must repeat the iteration process (even in the case where the piston motion is known in advance) to determine the relevant values at A.

Fig.13, Method-2



Here we consider the different way in which we do not use the characteristics. It depends on the idea of "decay of discontinuity" and "integral formula", by which Godunov's scheme also was constructed in the internal region.

First we write the integral formula for (I-1)

$$
\text{(I-3)} \qquad \oint u\,dx + v\,dt = 0 , \qquad \oint v\,dx + u\,dt = 0 .
$$

Hence we have the jump condition

$$[u]s + [v] = 0 ,$$

$$[v]s + [u] = 0 ,$$

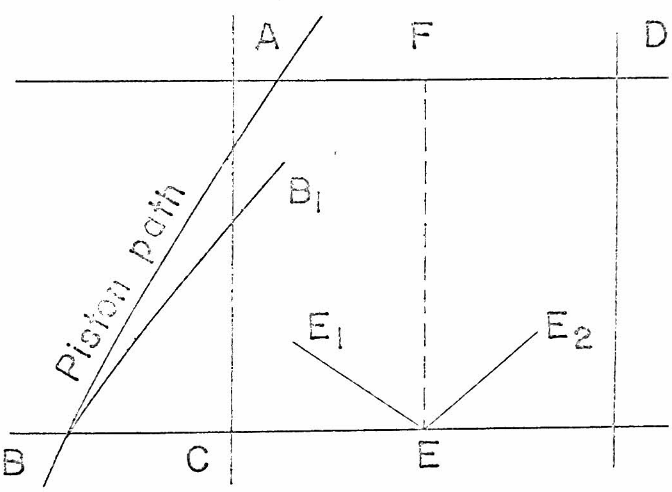


Fig.14-b

i.e.

$$
\text{(I-4)} \qquad s = \pm 1, \qquad [u] = \mp [v] ,
$$

where s is the slope of the discontinuity line, and $[\cdot]$ means the **jump of the quantity.** In this case the discontinuity line coincides with the characteristics.

We consider the case a and b :

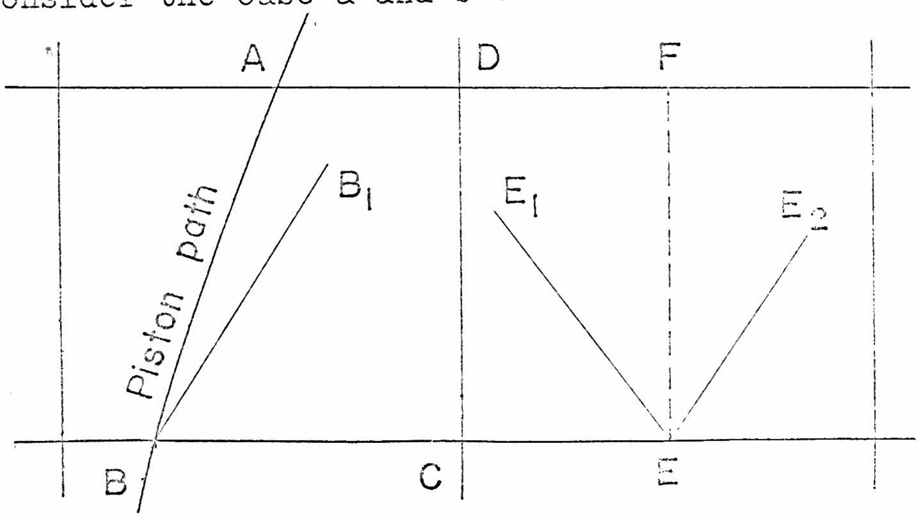


Fig.14-a

In both states we consider that the integrated mean values $\overline{u_{BE}}$, $\overline{v_{BE}}$ are kept constant between B and E, and that the mean value $u_M = \frac{1}{2}(u_A + u_B)$, $v_M = \frac{1}{2}(v_A + v_B)$ are kept constant between A and B. Then the discontinuity line $BB_1(s=1)$ is generated from the point B and along it we have by the jump condition (I-4)

$$u_M + v_M = \overline{u_{BE}} + \overline{v_{BE}} \quad .$$

From this formula v_M, and thus v_A are determined.

In the case a we take the integration path ABEFA and write down (I-3) in order to determine the state of D,

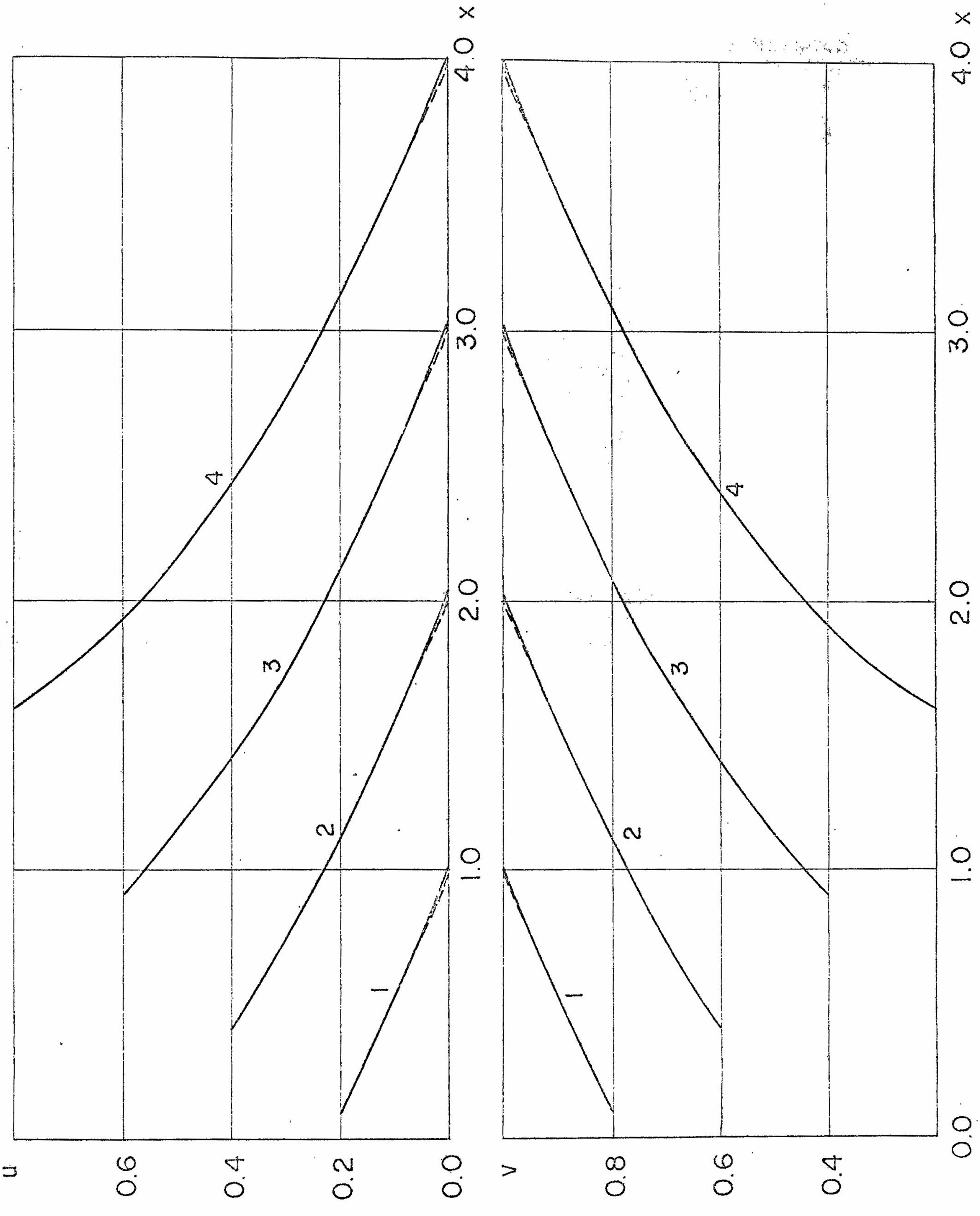
$$\widetilde{u}\overline{AF} = \overline{u_{BE}}\ \overline{BE} + \left\{ V_E - (u_M + v_M) \right\} \Delta t \quad ,$$

$$\widetilde{v}\overline{AF} = \overline{v_{BE}}\ \overline{BE} + \left\{ U_E - (u_M v_M + u_M) \right\} \Delta t \quad ,$$

where $\overline{AF}$ and $\overline{BE}$ mean the lengths of the intervals AF and BE respectively. The capital letters U_E and V_E mean the states in the region E_1EE_2 which are produced by the decay of the discontinuity at the point E and $\widetilde{u}$ and $\widetilde{v}$ are the mean values between A and F. Considering them as the values at the middle point of AF, we can calculate ones at D using also the values at A by interpolation or extrapolation. In the case b, the values at D are determined by (I-2). The experimental result by the method is shown in Fig.15 and is very good.

In order to compare the above three methods we calculated

Fig.15,   Method-3



the sum e of the absolute value of the differences between the exact **and the numerical solution** at mesh points on the right of the piston on each line t=const. These errors are shown in Fig.16.

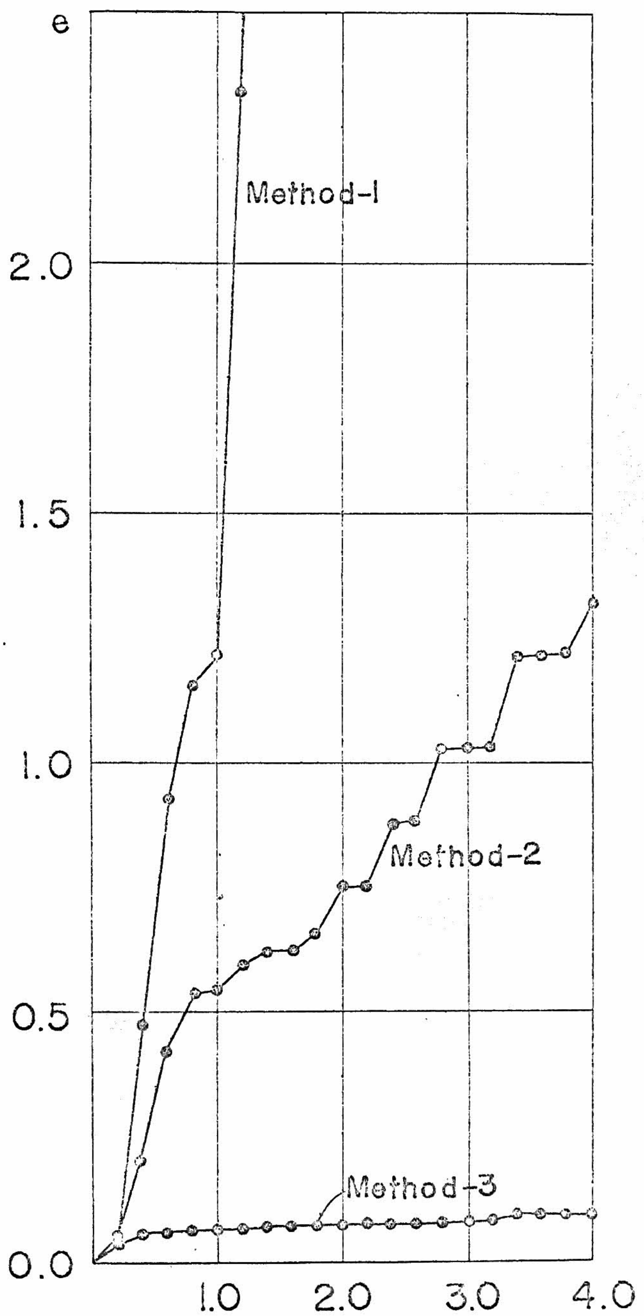


Fig.16, The evolution of the errors

Appendix II

Here we shall report the numerical expriments for the equation of fluid dynamics (2.1) in which the constantly accelerated piston motion (on the half way, and then constant-speed-motion) is treated by several methods. The piston path is given by the equation

$$x=\xi(t) = \begin{cases} \frac{1}{2}t^2 & 0 \leqslant t \leqslant 1 \\ t-\frac{1}{2} & 1 \leqslant t \end{cases} .$$

Initial conditions are $u(0,x)=0$, $\rho(0,x)=10.0$, $p(0,x)=2.0$.

(i) Method-1 We shall again consider two cases :

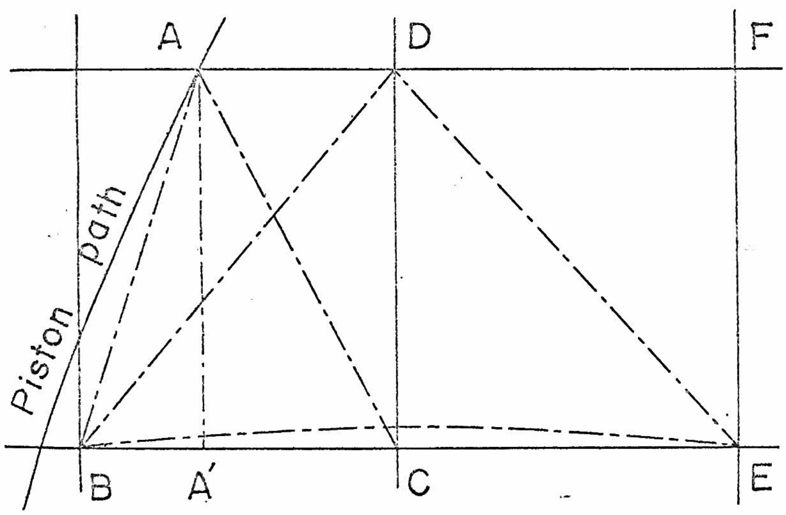


Fig.17-a

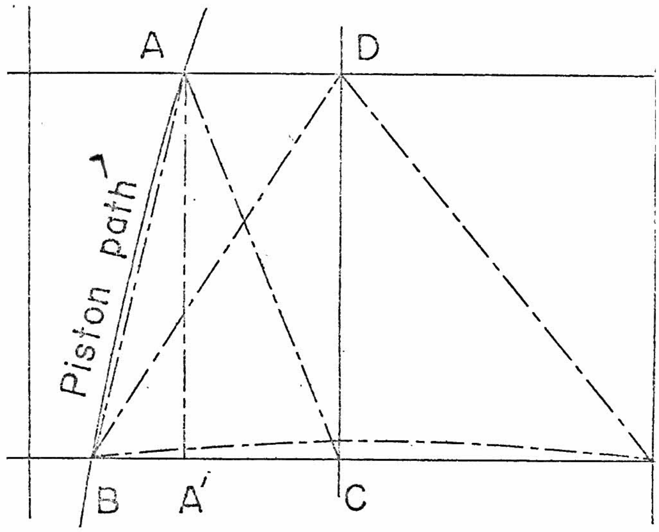


Fig.17-b

In both cases we use the formulas, fox example,

$$\frac{\rho_A-\rho_{A'}}{\Delta t} + \frac{(\rho u)_C-(\rho u)_B}{\overline{BC}} = 0 , \quad \rho_{A'} = \frac{\overline{BA'}\rho_C+\overline{A'C}\rho_B}{\overline{BC}} ,$$

$$\frac{\rho_D-\widetilde{\rho}_C}{\Delta t} + \frac{(\rho u)_E-(\rho u)_B}{\overline{BE}} = 0 , \quad \widetilde{\rho}_C = \frac{\overline{BC}\,\rho_E+\overline{CE}\,\rho_B}{\overline{BE}} .$$

Analogous construction is done for the rest equations.

The numerical result by this method is shown in the Fig.18.

(ii) Method-2 Instead of the last equation in the Method-1 we use the following formula,

$$\frac{\rho_D-\rho_C}{\Delta t} + \frac{(\rho u)_E-(\rho u)_B}{\overline{BE}} = 0 ,$$

see the Fig.19.

(iii) Method-3 This method is that of §2. Here we have put

$$\frac{p'_+ + p_+}{2} = \overline{p_+} , \qquad \text{(see(4.1))}$$

$$\frac{\rho'_+ + \rho_+}{2} = \overline{\rho_+} ,$$

where the prime means the valves at A.

The result by this method is shown in the Fig.20. The calculation overflowed and stopped halfway.

(iv) Method-4 This method also is that of §2. But we put

$$p'_+ = \overline{p_+} , \qquad \text{(see(4.1))}$$

$$\rho'_+ = \overline{\rho_+} .$$

Its results is shown in the Fig.21.

This result is best.

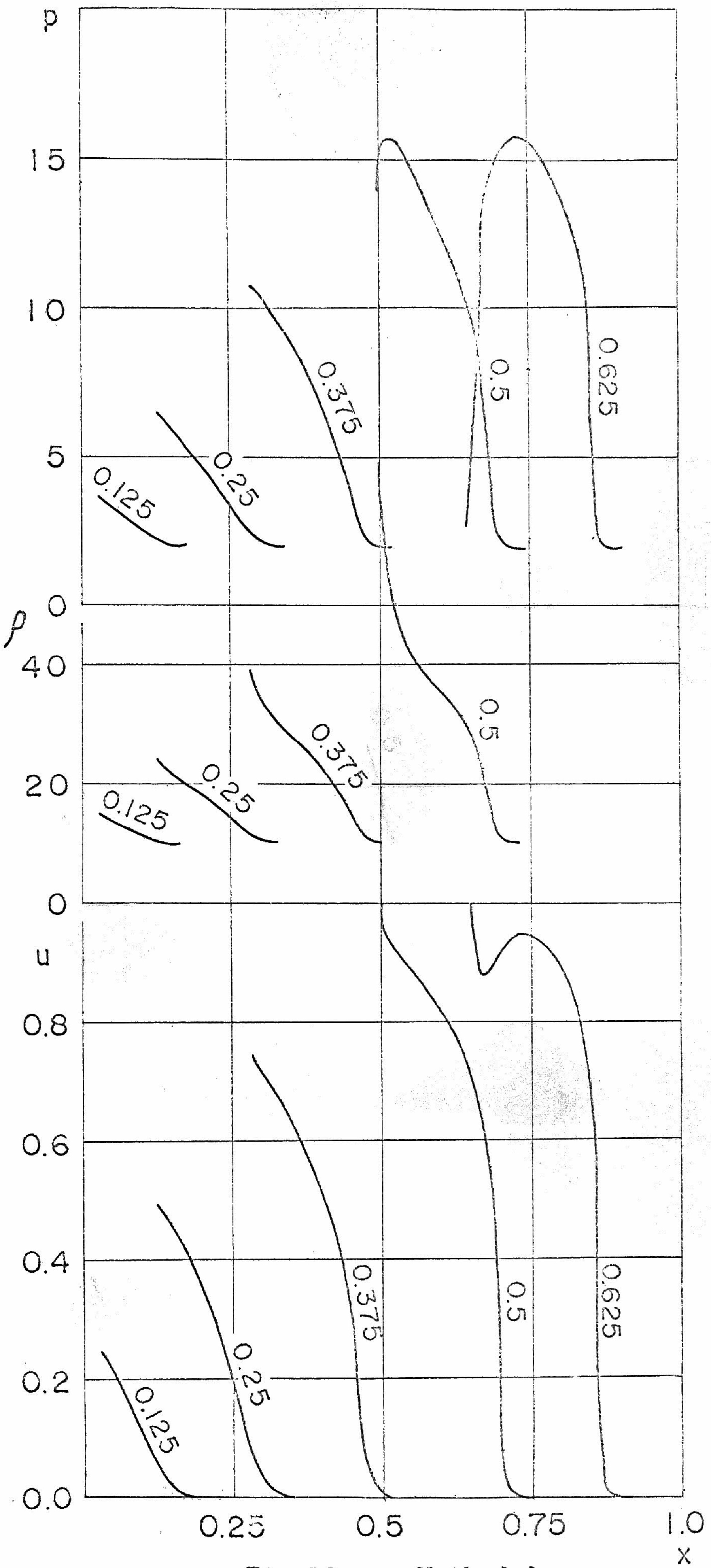


Fig.18, Method-1

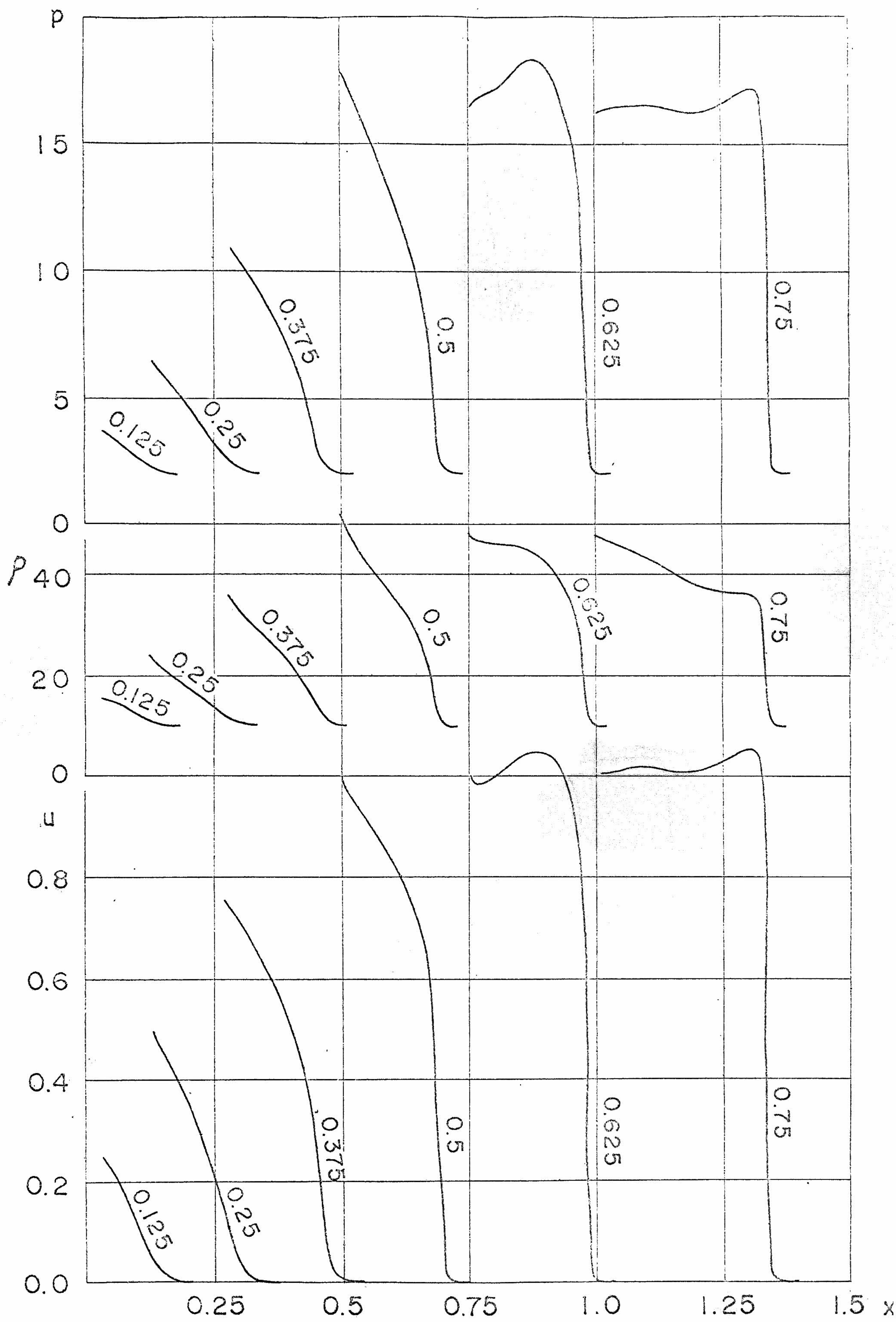


Fig.19, Method-2

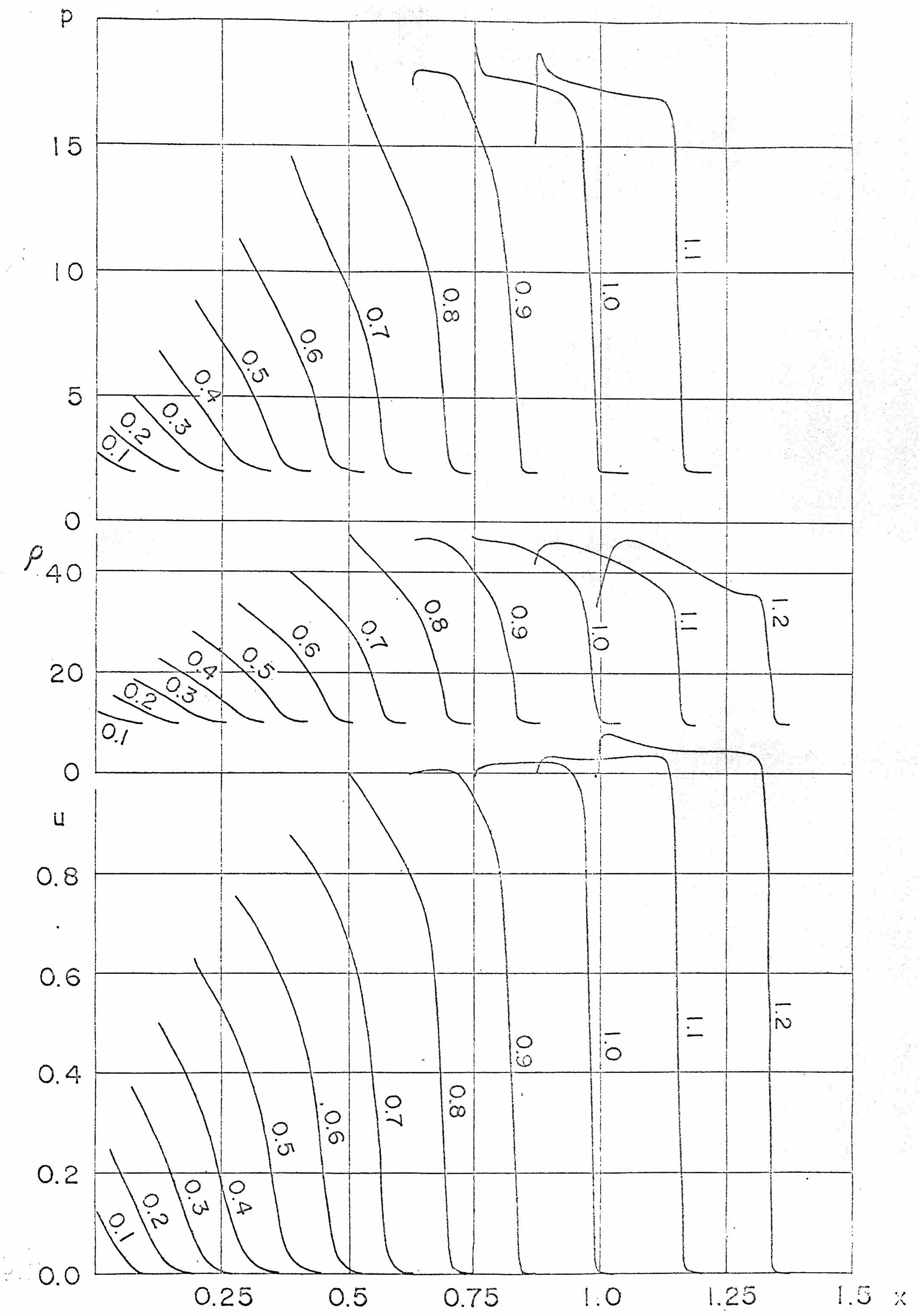


Fig.20, Method-3

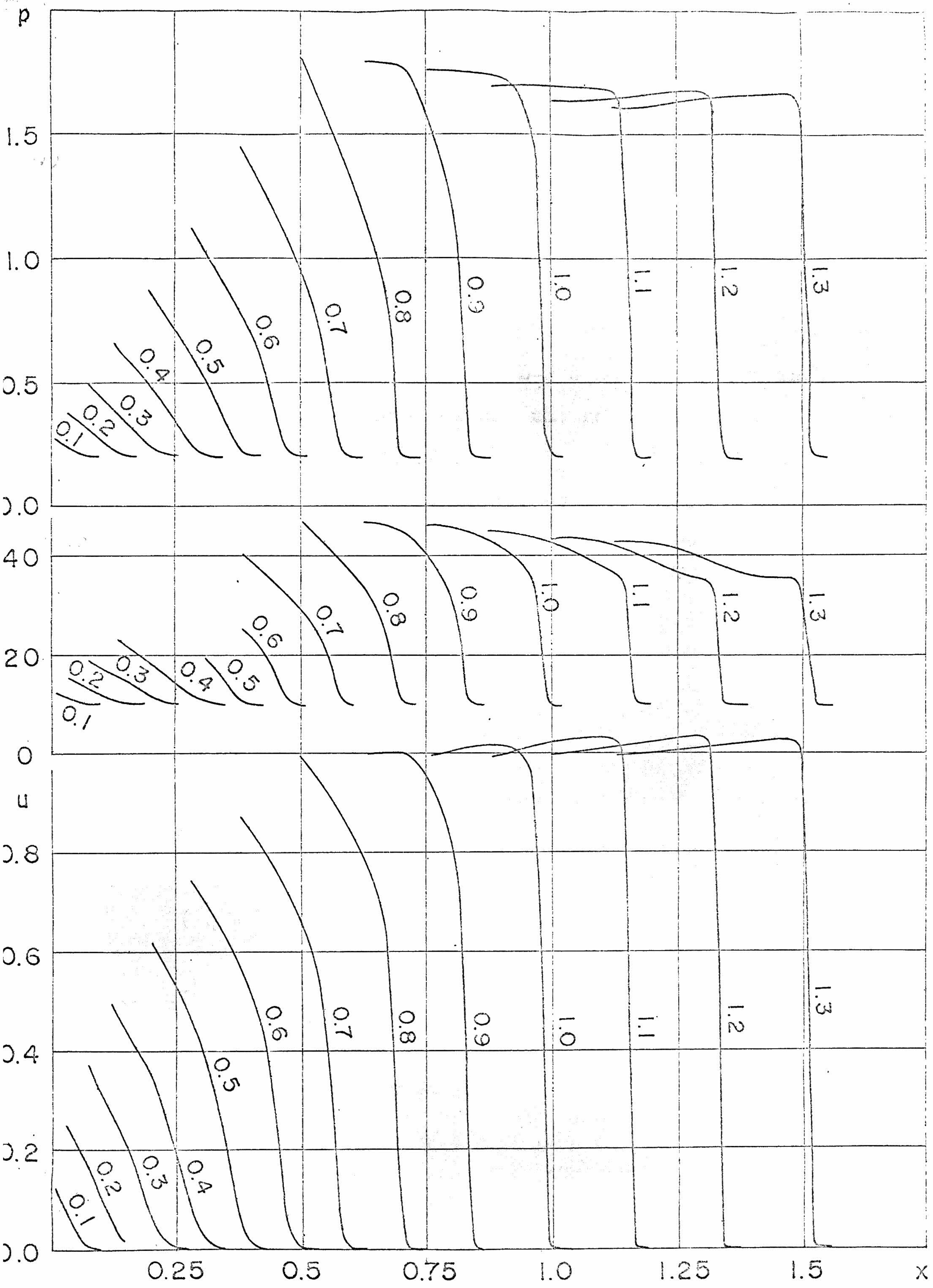


Fig.21, Method-4

(v) Method-5 In addition we shall report the calculation in which the Lax-Wendroff's viscosity method was used in the interior region. First in the neighbourhood of the piston we use the same method as Method-1. The result is shown in the Fig.22.

(vi) Method-6 As above we use the L-W's method. But in the neighbourhood of the piston we use the same method-2. The result is shown in the Fig.23.

Appendix III So far we considered the problem in the Eulerian form, but we can treat it also in the Lagrangian form. In this case the fundamental equation is

$$\frac{\partial V}{\partial t} = \frac{\partial u}{\partial q} ,$$

$$\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial q} ,$$

$$\frac{\partial (e + \frac{u^2}{2})}{\partial t} = -\frac{\partial pu}{\partial q} ,$$

$$(q = \int \rho dx , \quad \frac{\partial x}{\partial t} = u) .$$

Here q is the Lagrangian coordinate and V is the specific volume $(= \frac{1}{\rho})$. The other values are difined as in §2.

The equation of piston motion is given as follows :

$$\frac{du}{dt} = p(t,-0) - p(t,+0) , \quad \text{at} \quad q = 0.$$

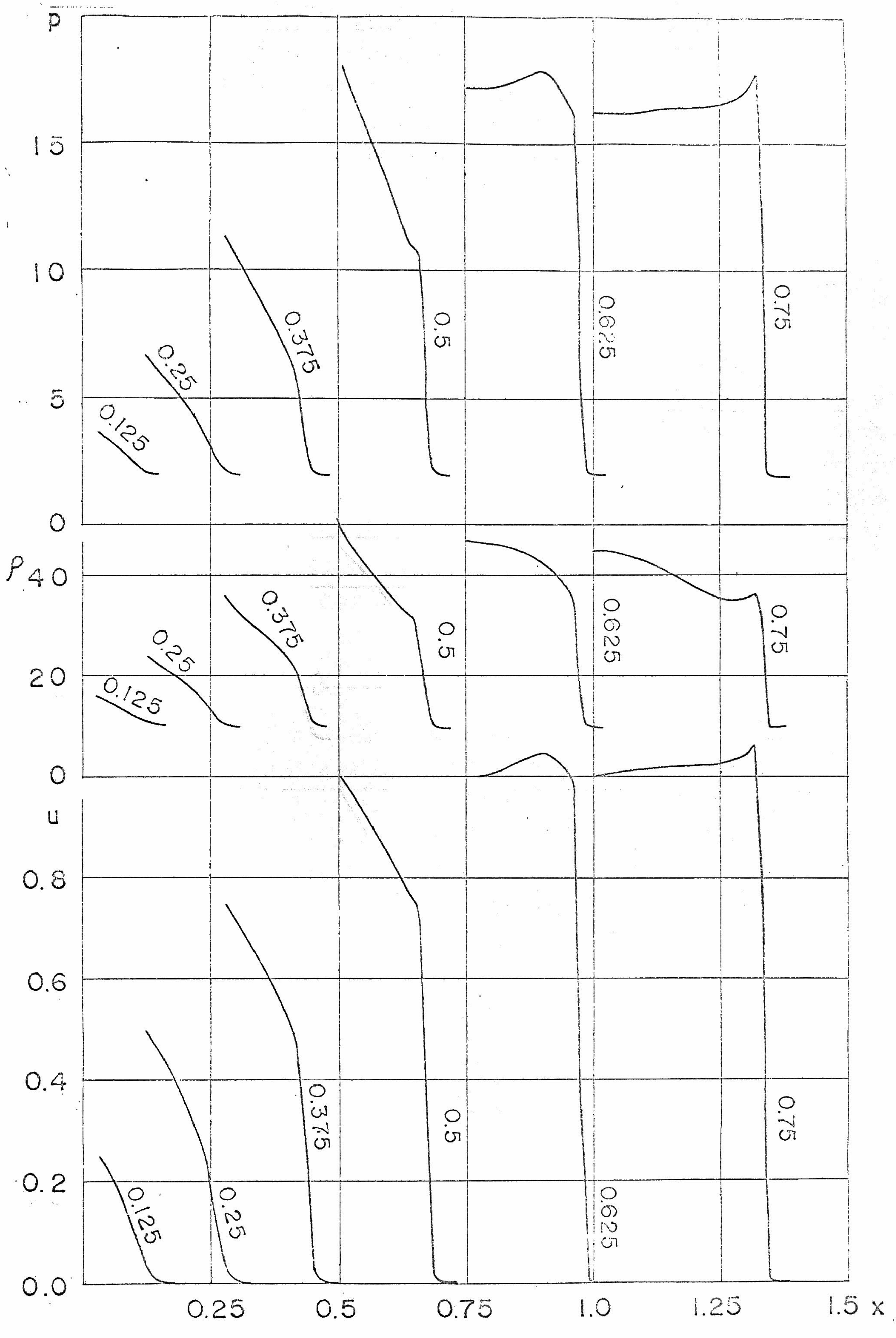


Fig.22, Method-5

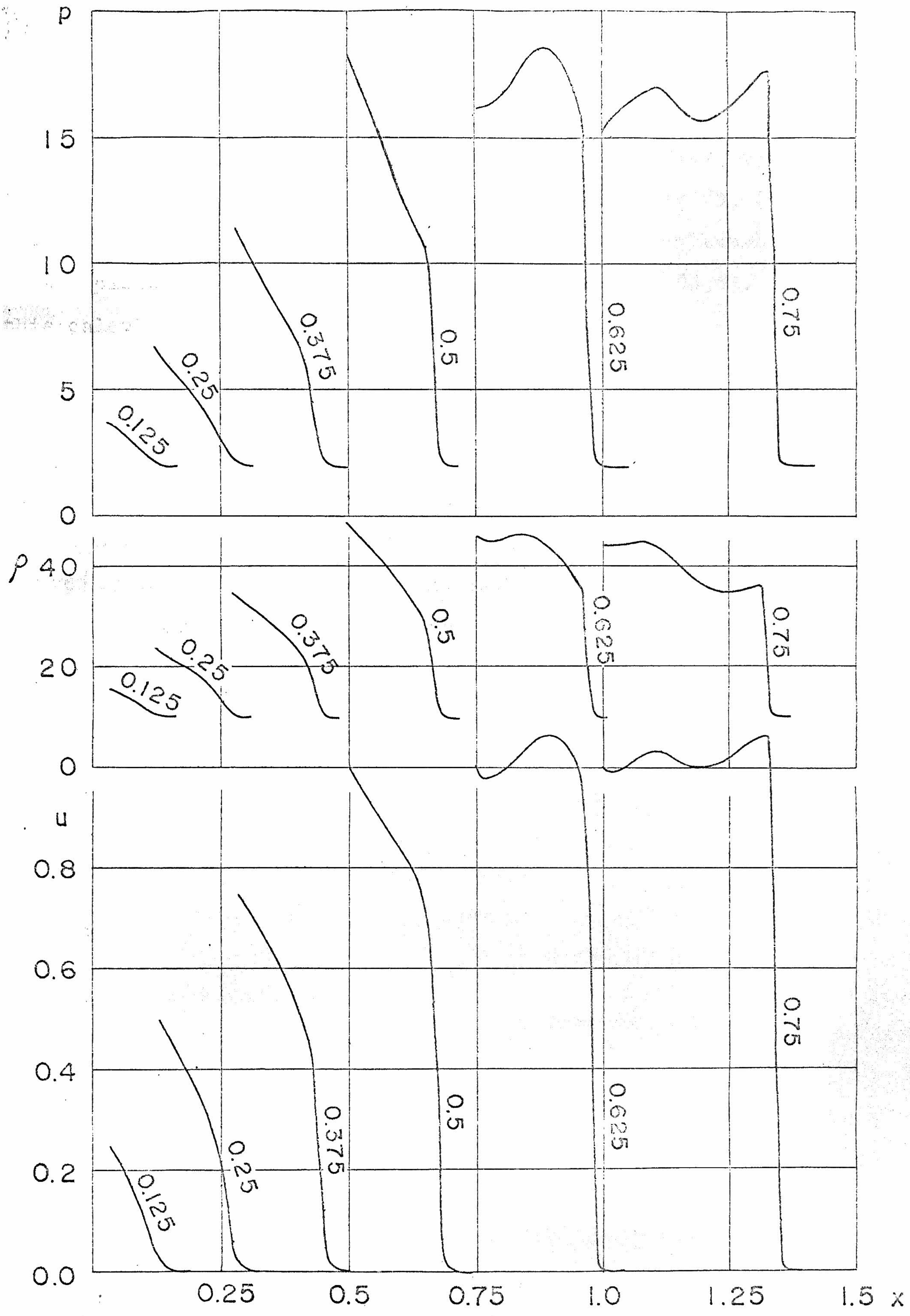


Fig.23, Method-6

We shall show the result in which the practical problem of §6 was solved by the exact Godunov's method. (see §3, in detail see [4]). The calculation method in the neighbourhoods of the piston and the wall is analogous to that in §4, §5. In this calculation the mesh width Δq in the left region to the piston is 0.7897 and that in the reght is 0.4387. The time interval Δt is 0.02.

The piston path and the pressure history at the wall are shown in Fig.6 and Fig.7, respectively, by the broken lines. This result seems to be closer to the experimental one than our result derived by the Eulerian form. But this comparison is not fair since the above mesh width Δq corresponds to the finer Δx.

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Part IV

A Difference Method for Boundary Value Problems with Derivative Boundary Conditions

Introduction

Pure difference methods for elliptic boundary value problems with derivative boundary conditions are treated by Batschelet[1], Giese[2], Lebedev[3-8], Volkov[9-10] and Wigley[11] etc.

For the same problems a kind of difference methods, what is called "Finite-element-method", are also investigated by Demjanovič[12], Friedrichs and Keller[13], Oganesyan[14-15], Oganesyan and Rukovetz[16-17] and Zlamál[18-19] etc. In this method a reduced minimal problem from the original boundary value problem is solved approximately in a subspace spanned by a class of finite number of "element" functions and their translated functions. The resulting difference scheme approximates automatically the differential equation in the interior of the domain and the boundary condition at points near the boundary. In these works the estimate of error between the exact and approximate solutions is given either in order of mesh width or precisely in an explicit form.

On the other hand, as far as we know, there were few works about difference methods for hyperbolic and parabolic mixed initial and boundary value problems with derivative boundary conditions in a domain of any shape. From mathematical interest we can refer to Lions[20] and Chekhlov[21] whose method is called "penalty method", in which the problems with homogeous mixed (Dirichlet and Neumann) boundary conditions are considered and are reduced to the problem of a differential equation with extended coefficients depending on the mesh width h over the region and with homogeneous Dirichlet conditions. But the rate of convergence is at most $O(\sqrt{h})$, which shows that this method is not fit for practical use.

Here we propose a difference method with rate of convergence $O(h)$ for mixed initial and boundary value problems of wave equation and heat equation with the boundary condition of third kind (and also for boundary value problems of elliptic equations) in a fairly arbitrary region on the plane. Our difference scheme corresponds to an integral formula of the original differential problem and has natural structures. The proof of convergence relies on the so called energy method. (cf. Ladyzhenskaya[22])

By trivial modification our method will be easily applied to the 3-dimensional case, to the equations with variable coefficients and to the problem with mixed boundary conditions.

§1. A mixed problem of a wave equation under a derivative boundary condition and its difference approximation

We consider the mixed problem of the wave equation in a cylindrical region $Q(T) = \Omega \times (0,T)$ in R^3 (Ω is a bounded domain in

$$(1.1) \qquad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(t,x,y) \qquad \text{in } Q(T)$$

under the initial conditions

$$(1.2) \qquad \begin{aligned} & u(0,x,y) = \varphi(x,y), \\ & \frac{\partial u}{\partial t}(0,x,y) = \Phi(x,y) \end{aligned}$$

and the boundary condition on the lateral surface

$$(1.3) \qquad \frac{\partial u}{\partial n} - \delta u = g(t,x,y).$$

Here $\frac{\partial}{\partial n}$ means the derivative along exterior normal to the boundary surface. δ is a constant. Under appropriate smoothness conditions of the boundary Γ of Ω and the functions f and g, as we know, a unique smooth solution exists. [23] Moreover we assume that at every point $P \in \Gamma$ there is a circle S such that $\overline{S} \cap \overline{\Omega} = P$.

For the sake of the future treatment we transform the equation (1.1) in an integral form by integrating the equation over any (t,x,y)-region $\omega \times [t,t+\Delta t]$ and using the Green's formula:

$$\iint_{\omega}\left[\frac{\partial u}{\partial t}(t+\Delta t,x,y)-\frac{\partial u}{\partial t}(t,x,y)\right]dxdy \tag{1.4}$$
$$=\int_t^{t+\Delta t}dt\int_{\partial\omega}\frac{\partial u}{\partial n}\,ds+\int_t^{t+\Delta t}dt\iint_{\omega}f\,dxdy\ ,$$

where ds means the line element of the boundary of ω.

Now we construct a net in R^2 whose nodes have coordinates of the form

$$x=mh,\quad y=nk\quad (m,n = 0,\ \pm1,\ \pm2,\ \cdots);$$

where h and k are distances between the two adjoining nodes in the x-direction and the y-direction respectively. Denote the set of all the nodes in Ω by Ω_h'.

We consider those nodes which adjoin to Ω_h'. We call a node adjoining to two nodes of Ω_h' a boundary mesh point of the first kind and call a node adjoining to one node of Ω_h' a boundary mesh point of the second kind. Now we draw the "broken" lines through half-integer points, but we erase a broken segment lying between a boundary mesh point of the second kind and the corresponding node of Ω_h'.

Here we supposed that h and k are so small that no nodes outside of Ω have three or more neighbouring nodes of Ω_h'. (It is possible under our assumption about Γ.) Then if there appears a node being contained in a triangular mesh whose sides consist of two broken segments and a part of the boundary of Ω, we count it in the class of boundary mesh points of the first kind, so that we need not consider original adjoining boundary mesh points of the second kind. We express the set of the left nodes of Ω_h' by Ω_h, which we call the set of interior mesh points.

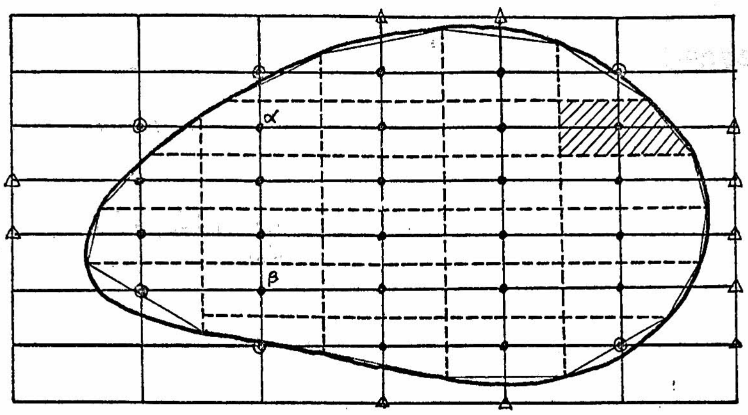

Fig.-1
The concerned mesh points

And we express the set of all the boundary mesh points by Γ_h. Finally we draw line segments connecting the two neighbouring intersecting points of the "broken" lines and the boundary , and then we have the polygonal region and express it by the same notation Ω_h. The polygon Ω_h consists of some triangle, quadrilateral and pentagonal meshes having a boundary of broken lines or a side of the polygon .

In order to construct a difference scheme, we apply the integral formula (1.4) over each quadrilateral or pentagonal mesh and approximate each term by a corresponding difference quotient as follows; for example, over the hatched mesh in Fig. 1 we have a fomula (after dividing by Δt)

$$(1.5) \qquad S_h u_{t\bar{t}} = (\bar{g}+\delta\{u\}_{x+h})\Delta\Gamma - a_{-h}u_{\bar{x}} + a_{+k}u_y - a_{-k}u_{\bar{y}} + S_h\bar{f}, \quad (*)$$

where $\Delta\Gamma$, a_{-h}, a_{+k} and a_{-k} are the length of the right, left, upper and lower side respectively, and S_h means the area of the mesh.

(*) $\{\ \}_{x+h}$ means that the quantity in the bracket is calculated at the boundary mesh point of second kind, while other terms are calculated at the concerned node.

For the forward and backward difference quotients we employed the following notations;

$$u_t = \frac{u(t+\Delta t,x,y)-u(t,x,y)}{\Delta t}, \quad u_{\bar{t}} = \frac{u(t,x,y)-u(t-\Delta t,x,y)}{\Delta t},$$

$$u_x = \frac{u(t,x+h,y)-u(t,x,y)}{h}, \quad u_{\bar{x}} = \frac{u(t,x,y)-(t,x-h,y)}{h},$$

$$u_y = \frac{u(t,x,y+k)-u(t,x,y)}{k}, \quad u_{\bar{y}} = \frac{u(t,x,y)-u(t,x,y-k)}{k}$$

and for difference quotients of second order, e.g.,

$$(u_t)_{\bar{t}} = u_{t\bar{t}} = u_{\bar{t}t}$$

$$= \frac{1}{\Delta t^2}\Big[u(t+\Delta t,x,y)-2u(t,x,y)+u(t-\Delta t,x,y)\Big].$$

$\bar{f}$ denotes the mean value of f over the concerned mesh and $\bar{g}$ denotes the mean value of g along the corresponding part of Γ.
If we determine the value of u at the boundary mesh point (x_0, y_0) of the second kind adjacent to the concerned mesh by the formula

$$a_{-h}u_{\bar{x}} = (\bar{g}+\delta u)\Delta\Gamma \quad \text{at } (x_0,y_0) \text{ or equivalently}$$

$$a_{+h}u_x = (\bar{g}+\delta\{u\}_{x+h})\Delta\Gamma \text{ at } (x_0-h,y_0),\ a_{-h} = a_{+h} = k,$$

(which, we note, is only the replacement of notation and itself does not mean the formal approximation of the boundary condition), the equation (1.5) takes the form

$$(1.6) \qquad S_h u_{t\bar{t}} = a_{+h}u_x - a_{-h}u_{\bar{x}} + a_{+k}u_y - a_{-k}u_{\bar{y}} + S_h\bar{f}.$$

We can use the above difference equation (1.6) at any quadrilateral mesh if we take

$$
(1.7)\qquad
\begin{aligned}
a_{-h}u_{\bar{x}} &= (\bar{g}+\delta u)\Delta\Gamma\ , & a_{-h} &= k && \text{on a right boundary mesh point,}\\
a_{+h}u_{x} &= -(\bar{g}+\delta u)\Delta\Gamma, & a_{+h} &= k && \text{on a left boundary mesh point,}\\
a_{-k}u_{\bar{y}} &= (\bar{g}+\delta u)\Delta\Gamma\ , & a_{-k} &= h && \text{on an upper boundary mesh point,}\\
a_{+k}u_{y} &= -(\bar{g}+\delta u)\Delta\Gamma, & a_{+k} &= h, && \text{on a lower boundary mesh point,}
\end{aligned}
$$

and $a_{+h} = a_{-h} = k$, $a_{+k} = a_{-k} = h$ on any interior mesh.(*)

For a pentagonal mesh we have a formula

$$
(1.8)\qquad S_h u_{t\bar{t}} = a_{+h}u_x - a_{-h}u_{\bar{x}} + a_{+k}u_y - a_{-k}u_{\bar{y}} + (\bar{g}+\delta u)\Delta\Gamma + S_h\bar{f},
$$

where a_{+h}, a_{-h}, a_{+k} and a_{-k} are the length of the right, left, upper and lower side respectively, and $\Delta\Gamma$ is that of the side of the polygon Ω_h. By using the function δ_h which equals unity on a pentagonal mesh and equals zero on a quadrilateral mesh, (1.6) and (1.8) can be written together in the form

$$
(1.9)\qquad S_h u_{t\bar{t}} = a_{+h}u_x - a_{-h}u_{\bar{x}} + a_{+k}u_y - a_{-k}u_{\bar{y}} + \delta_h(\bar{g}+\delta u)\Delta\Gamma + S_h\bar{f}.
$$

In order to determine the value of u at each boundary mesh point of the first kind facing a triangular mesh we apply one of the following formulae;

(*) From the construction of our net it is known that $\frac{a_{-h}}{h}$, $\frac{a_{+h}}{h}$, $\frac{a_{-k}}{k}$ and $\frac{a_{+k}}{k}$ are uniformly apart from zero and then $\frac{a_{-h}}{h} - \delta\Delta\Gamma$ etc. are not zero for sufficiently small h.

$$a_{-h}u_{\bar{x}}+a_{-k}u_{\bar{y}} = (\bar{g}+\delta u)\Delta\Gamma \qquad \text{at a right upper mesh point,}$$

$$a_{-h}u_{\bar{x}}-a_{+k}u_{y} = (\bar{g}+\delta u)\Delta\Gamma \qquad \text{at a right lower mesh point,}$$

$$(1.10) \qquad -a_{+h}u_{x}+a_{-k}u_{\bar{y}} = (\bar{g}+\delta u)\Delta\Gamma \qquad \text{at a left upper mesh point,}$$

and

$$-a_{+h}u_{x}-a_{+k}u_{y} = (\bar{g}+\delta u)\Delta\Gamma \qquad \text{at a left lower mesh point.}$$

At a boundary mesh point of the first kind facing a quadrilateral mesh we must apply another formula, for example, at a left lower point of Fig. 1 we use the formula

$$(1.11) \qquad -a_{+h}u_{x} + a_{-h}\{u_{\bar{x}}\}_{y+k} - a_{+k}u_{y} = (\bar{g}+\delta u)\Delta\Gamma,$$

where $\{\ \}_{y+k}$ means that the quantity in the bracket is calculated at $(x,y+k)$, while other terms are calculated at (x,y), and other notations are the same as above. At other boundary mesh points we can have analogous formulae. The formulae (1.10) and (1.11) approximate formally the boundary condition $\frac{\partial u}{\partial n}-\delta u=g$ with the error of order $O(h)+O(k)$, while (1.7) does not.

We also approximate the initial conditions (1.2) by the formulae

$$(1.12) \qquad u(0,x,y) = \varphi(x,y),$$

$$u(\Delta t,x,y) = \varphi(x,y)+\Delta t\psi(x,y).$$

Then we have the values of u on the planes $t = 0$ and $t = \Delta t$ by (1.12), and we can dermine the values of u on $t = 2\Delta t$, $3\Delta t, \ldots$ successively by using (1.9), (1.10) and (1.11) etc.

§2. Convergence of the scheme

Now we will prove that the solution of our difference scheme converges to the solution of the original problem (1.1),(1.2)and (1.3) under appropriate conditions.

We can rewrite the difference equation (1.9) in the form

$$S_h u_{t\bar{t}} = \frac{h}{2}(a_{+h}u_x)_{\bar{x}} + \frac{h}{2}(a_{-h}u_{\bar{x}})_x + \frac{k}{2}(a_{+k}u_y)_{\bar{y}} + \frac{k}{2}(a_{-k}u_{\bar{y}})_y + \\ + \delta_h(\bar{g}+\delta u)\Delta\Gamma + S_h\bar{f} \, ,$$

by using the fact that $a_{-h}(x) = a_{+h}(x-h)$ and $a_{-k}(y) = a_{+k}(y-k)$. We multiply the last equation by $(u_t + u_{\bar{t}})$ and transform it with the aid of the following formulae:

$$u_{t\bar{t}}(u_t + u_{\bar{t}}) = (u_{\bar{t}}^2)_t \, ,$$

$$(a_{+h}u_x)_{\bar{x}}(u_t+u_{\bar{t}}) = \left[a_{+h}u_x(u_t+u_{\bar{t}})\right]_{\bar{x}} - \left\{a_{+h}u_x(u_t+u_{\bar{t}})_x\right\}_{x-h} \, ,$$

$$(a_{-h}u_{\bar{x}})_x(u_t+u_{\bar{t}}) = \left[a_{-h}u_{\bar{x}}(u_t+u_{\bar{t}})\right]_x - \left\{a_{-h}u_{\bar{x}}(u_t+u_{\bar{t}})_{\bar{x}}\right\}_{x+h}$$

etc.,

where a curled bracket with a suffix have the same meaning as in (1.11).

Then we have the equation

$$S_h(u_{\bar t}^2)_t = \frac{h}{2}\left[a_{+h}u_x(u_t+u_{\bar t})\right]_{\bar x} + \frac{h}{2}\left[a_{-h}u_{\bar x}(u_t+u_{\bar t})\right]_x$$

$$+ \frac{k}{2}\left[a_{+k}u_y(u_t+u_{\bar t})\right]_{\bar y} + \frac{k}{2}\left[a_{-k}u_{\bar y}(u_t+u_{\bar t})\right]_y$$

$$- \frac{h}{2}\left\{a_{+h}u_x(u_t+u_{\bar t})_x\right\}_{x-h} - \frac{h}{2}\left\{a_{-h}u_{\bar x}(u_t+u_{\bar t})_{\bar x}\right\}_{x+h}$$

$$- \frac{k}{2}\left\{a_{+k}u_y(u_t+u_{\bar t})_y\right\}_{y-k} - \frac{k}{2}\left\{a_{-k}u_{\bar y}(u_t+u_{\bar t})_{\bar y}\right\}_{y+k}$$

$$+ \delta_h(\bar g + \delta u)(u_t+u_{\bar t})\Delta\Gamma + S_h\bar f(u_t+u_{\bar t}).$$

We multiply the last equation by Δt and sum over $\Omega_h \times \left[t=s\Delta t;\ s=1,2,\ldots p-1\right]$:

(2.2)
$$\sum_{\Omega_h} S_h u_{\bar t}^2 \Big|_{\Delta t}^{p\Delta t} =$$

$$= \sum_{s=1}^{p-1}\Delta t\Big[\frac{1}{2}\sum_{j}\sum_{i=i_G(j)} a_{+h}u_x(u_t+u_{\bar t}) - \frac{1}{2}\sum_{j}\sum_{i=i_L(j)-1} a_{+h}\, u_x(u_t+u_{\bar t})$$

$$+ \frac{1}{2}\sum_{j}\sum_{i=i_G(j)+1} a_{-h}u_{\bar x}(u_t+u_{\bar t}) - \frac{1}{2}\sum_{j}\sum_{i=i_L(j)} a_{-h}u_{\bar x}(u_t+u_{\bar t})$$

$$+ \frac{1}{2}\sum_{i}\sum_{j=j_G(i)} a_{+k}u_y(u_t+u_{\bar t}) - \frac{1}{2}\sum_{i}\sum_{j=j_L(i)-1} a_{+k}\, u_y(u_t+u_{\bar t})$$

$$+ \frac{1}{2}\sum_{i}\sum_{j=j_G(i)+1} a_{-k}u_{\bar y}(u_t+u_{\bar t}) - \frac{1}{2}\sum_{i}\sum_{j=j_L(i)} a_{-k}\, u_{\bar y}(u_t+u_{\bar t})$$

$$- \frac{1}{2}\sum_{\Omega_h} h\left\{a_{+h}u_x(u_t+u_{\bar t})_x\right\}_{x-h} - \frac{1}{2}\sum_{\Omega_h} h\left\{a_{-h}u_{\bar x}(u_t+u_{\bar t})_{\bar x}\right\}_{x+h}$$

$$- \frac{1}{2}\sum_{\Omega_h} k\left\{a_{+k}u_y(u_t+u_{\bar t})_y\right\}_{y-k} - \frac{1}{2}\sum_{\Omega_h} k\left\{a_{-k}u_{\bar y}(u_t+u_{\bar t})_{\bar y}\right\}_{y+k} +$$

$$+\sum_{\Gamma_h'}(g+\delta u)(u_t+u_{\bar t})\Delta\Gamma+\sum_{\Omega_h}S_h\bar f(u_t+u_{\bar t})\Big],$$

where $i_G(j)$, $i_L(j)$ — x-coordinate numbers of right and left ends of each row segment(*) of mesh points on $y=jk$ in Ω_h, and $j_G(i)$, $j_L(i)$ — y-coordinate numbers of upper and lower ends of each column segment of mesh points on $x=ih$ in Ω_h. $\sum_{\Gamma_h'}$ means $\sum_{\Gamma_h}\delta_h$. Here we note the following relations:

$$a_{+h}u_x(u_t+u_{\bar t})\Big|_{i_G}=a_{-h}u_{\bar x}(u_t+u_{\bar t})\Big|_{i_G+1}-ha_{+h}u_x(u_t+u_{\bar t})_x\Big|_{i_G},$$

$$a_{-h}u_{\bar x}(u_t+u_{\bar t})\Big|_{i_L}=a_{+h}u_x(u_t+u_{\bar t})\Big|_{i_L-1}+ha_{-h}u_x(u_t+u_{\bar t})_{\bar x}\Big|_{i_L},$$

$$a_{+k}u_y(u_t+u_{\bar t})\Big|_{j_G}=a_{-k}u_{\bar y}(u_t+u_{\bar t})\Big|_{j_G+1}-ka_{+k}u_y(u_t+u_{\bar t})_y\Big|_{j_G},$$

$$a_{-k}u_{\bar y}(u_t+u_{\bar t})\Big|_{j_L}=a_{+k}u_y(u_t+u_{\bar t})\Big|_{j_L-1}+ka_{-k}u_y(u_t+u_{\bar t})_{\bar y}\Big|_{j_L},$$

$$\sum_{\Omega_h}h\Big\{a_{+h}u_x(u_t+u_{\bar t})_x\Big\}_{x-h}+\sum_{\Omega_h}h\Big\{a_{-h}u_{\bar x}(u_t+u_{\bar t})_{\bar x}\Big\}_{x+h}$$
$$=\sum_{\Omega_h}ha_{+h}u_x(u_t+u_{\bar t})_x+\sum_{\Omega_h}ha_{-h}u_{\bar x}(u_t+u_{\bar t})_{\bar x}$$

and

$$\sum_{\Omega_h}k\Big\{a_{+k}u_y(u_t+u_{\bar t})_y\Big\}_{y-k}+\sum_{\Omega_h}k\Big\{a_{-k}u_{\bar y}(u_t+u_{\bar t})_{\bar y}\Big\}_{y+k}$$
$$=\sum_{\Omega_h}ka_{+k}u_y(u_t+u_{\bar t})_y+\sum_{\Omega_h}ka_{-k}u_{\bar y}(u_t+u_{\bar t})_{\bar y}\quad.$$

Applying these formulae to the right side of (2.2), we have

$$\sum_{\Omega_h}S_hu_{\bar t}^2\Big|_{\Delta t}^{p\Delta t}=$$
$$=\sum_{s=1}^{p-1}\Delta t\Big[\sum_{\substack{i=i_G(j)+1\\ j}}a_{-h}u_{\bar x}(u_t+u_{\bar t})-\sum_{\substack{i=i_L(j)-1\\ j}}a_{+h}u_x(u_t+u_{\bar t})+$$
$$+\sum_{\substack{j=j_G(i)+1\\ i}}a_{-k}u_{\bar y}(u_t+u_{\bar t})-\sum_{\substack{j=j_L(i)-1\\ i}}a_{+k}u_y(u_t+u_{\bar t})+$$

(*) If Ω is not convex, some mesh lines $y=jk$ may be divided into several segments by the boundary .

$$- \frac{1}{2}\Big(\sum_{\substack{i_G(j)\\ j}} + \sum_{\Omega_h} \Big) h\ a_{+h} u_x (u_t + u_{\bar t})_x - \frac{1}{2}\Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h} \Big) h\ a_{-h} u_{\bar x} (u_t + u_{\bar t})_{\bar x}$$

$$- \frac{1}{2}\Big(\sum_{\substack{j_G(i)\\ i}} + \sum_{\Omega_h} \Big) k\ a_{+k} u_y (u_t + u_{\bar t})_y - \frac{1}{2}\Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h} \Big) k\ a_{-k} u_{\bar y} (u_t + u_{\bar t})_{\bar y}$$

$$\left. + \sum_{\Gamma_h'} (\bar g + \delta u)(u_t + u_{\bar t}) \Delta\Gamma + \sum_{\Omega_h} S_h \bar f (u_t + u_{\bar t}) \right] .$$

Applying the boundary conditions (1.7),(1.10) and (1.11), we have

$$\sum_{\Omega_h} S_h u_{\bar t}^2 \Big|_{\Delta t}^{p\Delta t} =$$

$$= - \frac{1}{2} \sum_{s=1}^{p-1} \Delta t \Big[\Big(\sum_{\substack{i_G(j)\\ j}} + \sum_{\Omega_h} \Big) h\ a_{+h}\ u_x (u_t + u_{\bar t})_x + \Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h} \Big) h\ a_{-h} u_{\bar x} (u_t + u_{\bar t})_{\bar x} +$$

$$+ \Big(\sum_{\substack{j_G(i)\\ i}} + \sum_{\Omega_h} \Big) k\ a_{+k}\ u_y (u_t + u_{\bar t})_y + \Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h} \Big) k\ a_{-k} u_{\bar y} (u_t + u_{\bar t})_{\bar y} \Big]$$

$$+ \sum_{s=1}^{p-1} \Delta t \Big[\sum_{\Gamma_h} \Delta\Gamma (\bar g + \delta u)(u_t + u_{\bar t}) + \sum_{\Gamma_h'} \Delta\Gamma (\bar g + \delta u)(u_t + u_{\bar t}) +$$

$$+ \sum_{\Omega_h} S_h \bar f (u_t + u_{\bar t}) \Big] .$$

We transform each term of the right side as follows:

$$\sum_{s=1}^{p-1} \Delta t\ u_x (u_t + u_{\bar t})_x = \sum_{s=0}^{p-1} \Delta t \Big[u_x u_{xt} + \big\{ u_x u_{x\bar t} \big\}_{(s+1)\Delta t} \Big] - \Delta t\ u_x u_{xt} \Big|_0$$

$$- \Delta t\ u_x u_{x\bar t} \Big|_{p\Delta t}$$

$$= u_x^2 \Big|_0^{p\Delta t} - \Delta t\ u_x u_{x\bar t} \Big|_{p\Delta t} - \Delta t\ u_x u_{xt} \Big|_0 ,$$

$$\sum_{s=1}^{p-1} \Delta t\ u_{\bar x} (u_t + u_{\bar t})_{\bar x} = u_{\bar x}^2 \Big|_0^{p\Delta t} - \Delta t\ u_{\bar x} u_{\bar x \bar t} \Big|_{p\Delta t} - \Delta t\ u_{\bar x} u_{\bar x t} \Big|_0 ,$$

$$\sum_{s=1}^{p-1} \Delta t\ u_y (u_t + u_{\bar t})_y = u_y^2 \Big|_0^{p\Delta t} - \Delta t\ u_y u_{y\bar t} \Big|_{p\Delta t} - \Delta t\ u_y u_{yt} \Big|_0 ,$$

$$\sum_{s=1}^{p-1} \Delta t u_{\bar{y}}(u_t+u_{\bar{t}})_{\bar{y}} = u_{\bar{y}}^2\Big|_0^{p\Delta t} - \Delta t u_{\bar{y}} u_{\bar{y}\bar{t}}\Big|_{p\Delta t} - \Delta t u_{\bar{y}} u_{\bar{y}t}\Big|_0 ,$$

$$\sum_{s=1}^{p-1} \Delta t\ \bar{g}(u_t+u_{\bar{t}}) = \sum_{s=1}^{p-1} \Delta t\left[(u\bar{g})_t - \left\{u\bar{g}_{\bar{t}}\right\}_{(s+1)\Delta t} + (u\bar{g})_{\bar{t}} - \left\{u\bar{g}_t\right\}_{(s-1)\Delta t}\right]$$

$$= u\bar{g}\Big|_{\Delta t}^{p\Delta t} + u\bar{g}\Big|_0^{(p-1)\Delta t} - \sum_{s=2}^{p} \Delta t u\bar{g}_{\bar{t}} - \sum_{s=0}^{p-2} \Delta t u\bar{g}_t$$

$$= \left(\bar{g}(p\Delta t)+\bar{g}((p-1)\Delta t)\right)u(p\Delta t) - \Delta t\bar{g}((p-1)\Delta t)u_t((p-1)\Delta t)$$

$$- \sum_{s=2}^{p} \Delta t u\bar{g}_{\bar{t}} - \sum_{s=0}^{p-2} \Delta t u\bar{g}_t$$

$$-(\bar{g}(\Delta t)+\bar{g}(o))u(o) - \Delta t\bar{g}(\Delta t)u_{\bar{t}}'(\Delta t)$$

and

$$\sum_{s=1}^{p-1} \Delta t\delta u(u_t+u_{\bar{t}}) = \sum_{s=0}^{p-1} \Delta t\delta(u^2)_t - \Delta t\delta u(o)u_t(o) - \Delta t\delta u(p\Delta t)u_{\bar{t}}(p\Delta t)$$

$$= \delta u^2(p\Delta t) - \Delta t\delta u(p\Delta t)u_{\bar{t}}(p\Delta t)$$

$$-\delta u^2(o) - \Delta t\delta u(o)u_t(o).$$

Thus we have

$$(2.3) \qquad \sum_{\Omega_h} S_h u_{\bar{t}}^2\Big|_{\Delta t}^{p\Delta t}$$

$$= -\frac{1}{2}\Big[\Big(\sum_{\substack{i_G(j)\\ j}} + \sum_{\Omega_h}\Big) ha_{+h}u_x^2 + \Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h}\Big) ha_{-h}u_{\bar{x}}^2 +$$

$$+ \Big(\sum_{\substack{j_G(i)\\ i}} + \sum_{\Omega_h}\Big) ka_{+k}u_y^2 + \Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h}\Big) ka_{-k}u_{\bar{y}}^2\Big]\Big|_0^{p\Delta t}$$

$$+ \frac{\Delta t}{2}\Big[\Big(\sum_{\substack{i_G(j)\\ j}} + \sum_{\Omega_h}\Big) ha_{+h}u_x u_{x\bar{t}} + \Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h}\Big) ha_{-h}u_{\bar{x}}u_{\bar{x}\bar{t}} +$$

$$+ \Big(\sum_{\substack{j_G(i)\\ i}} + \sum_{\Omega_h}\Big) ka_{+k}u_y u_{y\bar{t}} + \Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h}\Big) ka_{-k}u_{\bar{y}}u_{\bar{y}\bar{t}}\Big]_{p\Delta t} +$$

$$+\frac{\Delta t}{2}\Big[\Big(\sum_{\substack{i_G(j)\\ j}}+\sum_{\Omega_h}\Big)ha_{+h}u_xu_{xt}+\Big(\sum_{\substack{i_L(j)\\ j}}+\sum_{\Omega_h}\Big)ha_{-h}u_{\bar{x}}u_{\bar{x}t}$$

$$+\Big(\sum_{\substack{j_G(i)\\ i}}+\sum_{\Omega_h}\Big)ka_{+k}u_yu_{yt}+\Big(\sum_{\substack{j_L(i)\\ i}}+\sum_{\Omega_h}\Big)ka_{-k}u_{\bar{y}}u_{\bar{y}t}\Big]_0$$

$$+\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\,\delta\,u^2\Big|_0^{p\Delta t}-\Delta t\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\,\delta\,u(p\Delta t)u_{\bar{t}}(p\Delta t)$$

$$-\Delta t\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\,\delta\,u(0)u_t(0)$$

$$+\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[(\bar{g}(p\Delta t)+\bar{g}((p-1)\Delta t))u(p\Delta t)-\Delta t\bar{g}((p-1)\Delta t)u_t((p-1)\Delta t)\Big]$$

$$-\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[(\bar{g}(\Delta t)+\bar{g}(0))u(0)-\Delta t\bar{g}(\Delta t)u_{\bar{t}}(\Delta t)\Big]$$

$$-\sum_{s=2}^{p}\Delta t\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma u\bar{g}_{\bar{t}}-\sum_{s=0}^{p-2}\Delta t\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)u\bar{g}_t$$

$$+\sum_{s=1}^{p-1}\Delta t\sum_{\Omega_h}S_h\bar{f}(u_t+u_{\bar{t}}).$$

Here we transform the second bracket of the right side as follows:

$$\frac{\Delta t}{2}\Big[\Big(\sum_{\substack{i_G(j)\\ j}}+\sum_{\Omega_h}\Big)ha_{+h}u_xu_{x\bar{t}}+\Big(\sum_{\substack{i_L(j)\\ j}}+\sum_{\Omega_h}\Big)ha_{-h}u_{\bar{x}}u_{\bar{x}\bar{t}}+$$

$$+\Big(\sum_{\substack{j_G(i)\\ i}}+\sum_{\Omega_h}\Big)ka_{+k}u_yu_{y\bar{t}}+\Big(\sum_{\substack{j_L(i)\\ i}}+\sum_{\Omega_h}\Big)ka_{-k}u_{\bar{y}}u_{\bar{y}\bar{t}}\Big]$$

$$=\Delta t\Big[\Big(\sum_{\substack{i_G(j)\\ j}}+\frac{1}{2}\sum_{\substack{i\neq i_G(j)\\ j}}\Big)a_{+h}u_x(u_{\bar{t}}(x+h)-u_{\bar{t}}(x))+\Big(\sum_{\substack{i_L(j)\\ j}}+\frac{1}{2}\sum_{\substack{i\neq i_L(j)\\ j}}\Big)a_{-h}u_{\bar{x}}(u_{\bar{t}}(x)-u_{\bar{t}}(x-h))$$

$$+\Big(\sum_{\substack{j_G(i)\\ i}}+\frac{1}{2}\sum_{\substack{j\neq j_G(i)\\ i}}\Big)a_{+k}u_y(u_{\bar{t}}(y+k)-u_{\bar{t}}(y))+\Big(\sum_{\substack{j_L(i)\\ i}}+\frac{1}{2}\sum_{\substack{j\neq j_L(i)\\ i}}\Big)a_{-k}u_{\bar{y}}(u_{\bar{t}}(y)-u_{\bar{t}}(y-k))\Big]$$

$$=\Delta t\Big[\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma(\bar{g}+\delta u)u_{\bar{t}}-$$

$$-\sum_{\substack{i_G(j)\\ j}}a_{+h}u_xu_{\bar{t}}+\sum_{\substack{i_L(j)\\ j}}a_{-h}u_{\bar{x}}u_{\bar{t}}-\sum_{\substack{j_G(i)\\ i}}a_{+k}u_yu_{\bar{t}}+\sum_{\substack{j_L(i)\\ i}}a_{-k}u_{\bar{y}}u_{\bar{t}}+$$

$$+\frac{1}{2}\sum_{\substack{i\neq i_G(j)\\ j}}a_{+h}u_x(u_{\bar{t}}(x+h)-u_{\bar{t}}(x))+\frac{1}{2}\sum_{\substack{i\neq i_L(j)\\ j}}a_{-h}u_{\bar{x}}(u_{\bar{t}}(x)-u_{\bar{t}}(x-h))+$$

$$+\frac{1}{2}\sum_{\substack{j\neq j_G(i)\\ i}} a_{+k}u_y(u_{\bar{t}}(y+k)-u_{\bar{t}}(y))+\frac{1}{2}\sum_{\substack{j\neq j_L(i)\\ i}} a_{-k}u_{\bar{y}}(u_{\bar{t}}(y)-u_{\bar{t}}(y-k))\Big] .$$

By the last relation and intial conditions (1.12), the equation (2.3) takes the form

$$(2.4)\qquad \sum_{\Omega_h} S_h u_{\bar{t}}^2(p\Delta t)+\frac{1}{2}\Big[\Big(\sum_{\substack{i_G(j)\\ j}}+\sum_{\Omega_h}\Big) ha_{+h}u_x^2+\Big(\sum_{\substack{i_L(j)\\ j}}+\sum_{\Omega_h}\Big) ha_{-h}u_{\bar{x}}^2+$$

$$+\Big(\sum_{\substack{j_G(i)\\ i}}+\sum_{\Omega_h}\Big) ka_{+k}u_y^2+\Big(\sum_{\substack{j_L(i)\\ i}}+\sum_{\Omega_h}\Big) ka_{-k}u_{\bar{y}}^2\Big]_{p\Delta t}$$

$$=\Delta t\Big[-\sum_{\substack{i_G(j)\\ j}} a_{+h}u_xu_{\bar{t}}+\sum_{\substack{i_L(j)\\ j}} a_{-h}u_{\bar{x}}u_{\bar{t}}-\sum_{\substack{j_G(i)\\ i}} a_{+k}u_yu_{\bar{t}}+\sum_{\substack{j_L(i)\\ i}} a_{-k}u_{\bar{y}}u_{\bar{t}}$$

$$+\frac{1}{2}\sum_{\substack{i\neq i_G(j)\\ j}} a_{+h}u_x(u_{\bar{t}}(x+h)-u_{\bar{t}}(x))+\frac{1}{2}\sum_{\substack{i\neq i_L(j)\\ j}} a_{-h}u_{\bar{x}}(u_{\bar{t}}(x)-u_{\bar{t}}(x-h))$$

$$+\frac{1}{2}\sum_{\substack{j\neq j_G(i)\\ i}} a_{+k}u_y(u_{\bar{t}}(y+k)-u_{\bar{t}}(y))+\frac{1}{2}\sum_{\substack{j\neq j_G(i)\\ i}} a_{-k}u_{\bar{y}}(u_{\bar{t}}(y)-u_{\bar{t}}(y-k))\Big]_{p\Delta t}$$

$$+\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[\delta u^2-\Delta t\delta u u_{\bar{t}}\Big]_{p\Delta t}$$

$$+\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[(\bar{g}(p\Delta t)+\bar{g}((p-1)\Delta t))u(p\Delta t)-\Delta t\bar{g}((p-1)\Delta t)u_{\bar{t}}(p\Delta t)\Big]$$

$$-\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[\sum_{s=2}^{p}\Delta t u\bar{g}_{\bar{t}}+\sum_{s=0}^{p-2}\Delta t u\bar{g}_t\Big]$$

$$+\sum_{s=1}^{p-1}\Delta t\sum_{\Omega_h} S_h\bar{f}(u_t+u_{\bar{t}})+\sum_{\Omega_h} S_h\psi^2$$

$$+\frac{1}{2}\Big[\Big(\sum_{\substack{i_G(j)\\ j}}+\sum_{\Omega_h}\Big) ha_{+h}(\varphi_x^2+\Delta t\varphi\psi_x)+\Big(\sum_{\substack{i_L(j)\\ j}}+\sum_{\Omega_h}\Big) ha_{-h}(\varphi_{\bar{x}}^2+\Delta t\varphi\psi_{\bar{x}})$$

$$+\Big(\sum_{\substack{j_G(i)\\ i}}+\sum_{\Omega_h}\Big) ka_{+k}(\varphi_y^2+\Delta t\varphi\psi_y)+\Big(\sum_{\substack{j_L(i)\\ i}}+\sum_{\Omega_h}\Big) ka_{-k}(\varphi_{\bar{y}}^2+\Delta t\varphi\psi_{\bar{y}})\Big]$$

$$-\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[(\bar{g}(\Delta t)+\bar{g}(0))\varphi-\Delta t\bar{g}(\Delta t)\psi\Big] .$$

Each term in the right side of the last equation can be estimated as follows ;

$$\left| \Delta t \left[-\sum_{\substack{i_G(j)\\ j}} a_{+h} u_x u_{\bar t} + \sum_{\substack{i_L(j)\\ j}} a_{-h} u_{\bar x} u_{\bar t} - \sum_{\substack{j_G(i)\\ i}} a_{+k} u_y u_{\bar t} + \sum_{\substack{j_L(i)\\ i}} a_{-k} u_{\bar y} u_{\bar t} \right] \right|$$

$$\leqq \Delta t \left[\sum_{\substack{i_G(j)\\ j}} a_{+h} \left(\frac{u_{\bar t}^2}{8} + 2u_x^2 \right) + \sum_{\substack{i_L(j)\\ j}} a_{-h} \left(\frac{u_{\bar t}^2}{8} + 2u_{\bar x}^2 \right) + \sum_{\substack{j_G(i)\\ i}} a_{+k} \left(\frac{u_{\bar t}^2}{8} + 2u_y^2 \right) + \sum_{\substack{j_L(i)\\ i}} a_{-k} \left(\frac{u_{\bar t}^2}{8} + 2u_{\bar y}^2 \right.\right.$$

$$\left| \frac{\Delta t}{2} \left[\sum_{\substack{i \neq i_G(j)\\ j}} a_{+h} u_x \left(u_{\bar t}(x+h) - u_{\bar t}(x) \right) + \sum_{\substack{i \neq i_L(j)\\ j}} a_{-h} u_{\bar x} \left(u_{\bar t}(x) - u_{\bar t}(x-h) \right) + \right.\right.$$

$$\left.\left. + \sum_{\substack{j \neq j_G(i)\\ i}} a_{+k} u_y \left(u_{\bar t}(y+k) - u_{\bar t}(y) \right) + \sum_{\substack{j \neq j_L(i)\\ i}} a_{-k} u_{\bar y} \left(u_{\bar t}(y) - u_{\bar t}(y-k) \right) \right] \right|$$

$$\leqq \frac{\Delta t}{2} \left[\sum_{\substack{i \neq i_G(j)\\ j}} a_{+h} \left(\frac{u_{\bar t}(x+h)^2 + u_{\bar t}(x)^2}{4} + 2u_x^2 \right) + \sum_{\substack{i \neq i_L(j)\\ j}} a_{-h} \left(\frac{u_{\bar t}(x)^2 + u_{\bar t}(x-h)^2}{4} + 2u_{\bar x}^2 \right) \right.$$

$$\left. + \sum_{\substack{j \neq j_G(i)\\ i}} a_{+k} \left(\frac{u_{\bar t}(y+k)^2 + u_{\bar t}(y)^2}{4} + 2u_y^2 \right) + \sum_{\substack{j \neq j_L(i)\\ i}} a_{-k} \left(\frac{u_{\bar t}(y)^2 + u_{\bar t}(y-k)^2}{4} + 2u_{\bar y}^2 \right) \right]$$

$$= \frac{\Delta t}{2} \left[\sum_{\substack{i = i_G(j)\\ j}} \frac{a_{+h} + a_{-h}}{4} u_{\bar t}^2 + \sum_{\substack{i = i_L(j)\\ j}} \frac{a_{+h} + a_{-h}}{4} u_{\bar t}^2 + \sum_{\substack{i \neq i_G(j),\, i_L(j)\\ j}} \frac{a_{+h} + a_{-h}}{2} u_{\bar t}^2 \right.$$

$$+ \sum_{\substack{j = j_G(i)\\ i}} \frac{a_{+k} + a_{-k}}{4} u_{\bar t}^2 + \sum_{\substack{j = j_L(i)\\ i}} \frac{a_{+k} + a_{-k}}{4} u_{\bar t}^2 + \sum_{\substack{j \neq j_G(i),\, j_L(i)\\ i}} \frac{a_{+k} + a_{-k}}{2} u_{\bar t}^2$$

$$\left. + \sum_{\substack{i \neq i_G(j)\\ j}} 2a_{+h} u_x^2 + \sum_{\substack{i \neq i_L(j)\\ j}} 2a_{-h} u_{\bar x}^2 + \sum_{\substack{j \neq j_G(i)\\ i}} 2a_{+k} u_y^2 + \sum_{\substack{j \neq j_L(i)\\ i}} 2a_{-k} u_{\bar y}^2 \right] ,$$

$$\left| \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \Delta\Gamma \left(\bar{g}(p\Delta t)+\bar{g}((p-1)\Delta t) \right) u(p\Delta t) \right|$$

$$\leq \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \Delta\Gamma \left[u(p\Delta t)^2 + \frac{1}{2}\left(\bar{g}(p\Delta t)^2 + \bar{g}((p-1)\Delta t)^2 \right) \right] .$$

And by using the boundary condition we have

$$\left| \Delta t \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \Delta\Gamma \; \bar{g}((p-1)\Delta t) u_{\bar{t}}(p\Delta t) \right|$$

$$\leq \frac{1}{2} \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \Delta\Gamma \left[\bar{g}((p-1)\Delta t)^2 + \Delta t^2 u_{\bar{t}}(p\Delta t)^2 \right]$$

$$\leq C_1 \Delta t \sum_{\Omega_h} S_h \, u_{\bar{t}}(p\Delta t)^2 + \frac{1}{2} \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \Delta\Gamma \left[\bar{g}((p-1)\Delta t)^2 + C_2 \sqrt{hk} \Delta t^2 \bar{g}_{\bar{t}}^2 \right],$$

where C_1 and C_2 are constants independent of Δt, h and k. Other terms in the right side of (2.4) are also estimated by Schwarz's inequality. Then we have the following inequality.

$$(2.5) \quad \sum_{\Omega_h} S_h u_{\bar{t}}(p\Delta t)^2 + \frac{1}{2}\Bigg[\Big(\sum_{\substack{i_G(j)\\ j}} + \sum_{\Omega_h} \Big) h a_{+h} u_x^2 + \Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h} \Big) h a_{-h} u_{\bar{x}}^2 + $$

$$+ \Big(\sum_{\substack{j_G(i)\\ i}} + \sum_{\Omega_h} \Big) k a_{+k} u_y^2 + \Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h} \Big) k a_{-k} u_{\bar{y}}^2 \Bigg]_{p\Delta t} \leq$$

$$
\begin{aligned}
&\leq \Delta t \Big[\sum_{\substack{i_G(j)\\ j}} \Big(\frac{a_{+h}}{4} + \frac{a_{-h}}{8}\Big) u_{\bar t}^2 + \sum_{\substack{i_L(j)\\ j}} \Big(\frac{a_{-h}}{4} + \frac{a_{+h}}{8}\Big) u_{\bar t}^2 + \sum_{\substack{j_G(i)\\ i}} \Big(\frac{a_{+k}}{4} + \frac{a_{-k}}{8}\Big) u_{\bar t}^2 \\
&\quad + \sum_{\substack{j_L(i)\\ i}} \Big(\frac{a_{-k}}{4} + \frac{a_{+k}}{8}\Big) u_{\bar t}^2 + \sum_{\substack{i \neq i_G(j),\, i_L(j)\\ j}} \frac{a_{+h} + a_{-h}}{4} u_{\bar t}^2 + \sum_{\substack{j \neq j_G(i),\, j_L(i)\\ i}} \frac{a_{+k} + a_{-k}}{4} u_{\bar t}^2 \\
&\quad + \sum_{\substack{i_G(j)\\ j}} 2a_{+h} u_x^2 + \sum_{\substack{i_L(j)\\ j}} 2a_{-h} u_{\bar x}^2 + \sum_{\substack{j_G(i)\\ i}} 2a_{+k} u_y^2 + \sum_{\substack{j_L(i)\\ i}} 2a_{-k} u_{\bar y}^2 \\
&\quad + \sum_{\substack{i \neq i_G(j)\\ j}} a_{+h} u_x^2 + \sum_{\substack{i \neq i_L(j)\\ j}} a_{-h} u_{\bar x}^2 + \sum_{\substack{j \neq j_G(i)\\ i}} a_{+k} u_y^2 + \sum_{\substack{j \neq j_L(i)\\ i}} a_{-k} u_{\bar y}^2 \Big]_{p\Delta t} \\
&\quad + C_1 \Delta t \sum_{\Omega_h} S_h u_{\bar t} (p\Delta t)^2 + \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h u_{\bar t}^2 \\
&\quad + \Big(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\Big) \Delta\Gamma \Big[\Big(1 + \frac{3}{2}|\delta|\Big) u^2 + \Big(\frac{\Delta t^2}{2}|\delta| + \frac{\Delta t^2}{2}\Big) u_{\bar t}^2 \Big]_{p\Delta t} \\
&\quad + \sum_{s=0}^{p-1} \Delta t \Big(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\Big) u\, (s\Delta t)^2 \Delta\Gamma \\
&\quad + \Big(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\Big) \Delta\Gamma \Big[2\bar g((p-1)\Delta t)^2 + \Delta t^2 \Big(1 + \frac{C_2\sqrt{hk}}{2}\Big) g_{\bar t}(p\Delta t)^2 \\
&\quad + \sum_{s=0}^{p-1} \Delta t \Big(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\Big) \Delta\Gamma\, \bar g_t^2 + \sum_{s=1}^{p-1} \Delta t \sum_{\Omega_h} S_h f^2 + \sum_{\Omega_h} S_h \psi^2 \\
&\quad + \frac{1}{2} \Big(\sum_{\substack{i_G(j)\\ j}} + \sum_{\Omega_h}\Big) h a_{+h} \Big[\Big(1 + \frac{\Delta t}{2}\Big) \varphi_x^2 + \frac{\Delta t}{2} \psi_x^2 \Big] + \Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h}\Big) h a_{-h} \Big[\Big(1 + \frac{\Delta t}{2}\Big) \varphi_{\bar x}^2 + \frac{\Delta t}{2} \psi_{\bar x}^2 \Big] \\
&\qquad + \Big(\sum_{\substack{j_G(i)\\ i}} + \sum_{\Omega_h}\Big) k a_{+k} \Big[\Big(1 + \frac{\Delta t}{2}\Big) \varphi_y^2 + \frac{\Delta t}{2} \psi_y^2 \Big] + \Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h}\Big) k a_{-k} \Big[\Big(1 + \frac{\Delta t}{2}\Big) \varphi_{\bar y}^2 + \frac{\Delta t}{2} \psi_{\bar y}^2 \Big] \Big\} \\
&\quad + \frac{1}{2} \Big(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\Big) \Delta\Gamma \Big[\bar g(0)^2 + (1 + \Delta t) \bar g(\Delta t)^2 + \varphi^2 + \Delta t \psi^2 \Big] .
\end{aligned}
$$

Here we note two lemmas.

Lemma 1 For any u defined on $\Omega_h + T_h$, we have the inequality

$$\sum_{T_h} u^2 \Delta\Gamma + \sum_{T_h'} u^2 \Delta\Gamma$$

$$\leqq \varepsilon \sum_{\Omega_h} \left[h(a_{+h} u_x^2 + a_{-h} u_{\bar{x}}^2) + k(a_{+k} u_y^2 + a_{-k} u_{\bar{y}}^2) \right] + C(\varepsilon) \sum_{\Omega_h} S_h u^2$$

where ε is an arbitrarily small constant. The constant $C(\varepsilon)$ depends on ε and becomes larger as ε becomes smaller.

Proof of Lemma 1. We first note that for any v

$$(2.6)\quad \sum_{\substack{i_G(j)+1 \\ j}} a_{+h}(x-h)v - \sum_{\substack{i_L(j)-1 \\ j}} a_{+h}(x)v = \sum_{\Omega_h + \{i_L(j)-1\}} h a_{+h} v_x + \sum_{\Omega_h} (a_{+h}(x) - a_{+h}(x-h))\, v,$$

where $\sum_{\Omega_h + \{i_L(j)-1\}}$ means the summation over all the mesh points of Ω_h and all the boundary mesh points having the coordinate numbers $i_L(j)-1$. In fact,

$$\sum_{\Omega_h+\{i_L(j)-1\}} h a_{+h} v_x$$

$$= \sum_j \Big[a_{+h}(i_G)\big(v(i_G+1)-v(i_G)\big) + a_{+h}(i_G-1)\big(v(i_G)-v(i_G-1)\big) +$$

$$+\cdots+ a_{+h}(i_L)\big(v(i_L+1)-v(i_L)\big) + a_{+h}(i_L-1)\big(v(i_L)-v(i_L-1)$$

$$= \sum_j a_{+h}(i_G)v(i_G+1) - \sum_j a_{+h}(i_L-1)v(i_L-1)$$

$$-\sum_j \Big[\big(a_{+h}(i_G)-a_{+h}(i_G-1)\ v(i_G) + \big(a_{+h}(i_G-1)-a_{+h}(i_G-2)\ v(i_G-1) +$$

$$+\cdots+\big(a_{+h}(i_L)-a_{+h}(i_L-1)\big)v(i_L)\Big]$$

$$= \sum_{\substack{i_G(j)+1\\ j}} a_{+h}(x-h)v(x) - \sum_{\substack{i_L(j)-1\\ j}} a_{+h}(x)v(x) - \sum_{\Omega_h}\big(a_{+h}(x)-a_{+h}(x-h)\big)v(x)$$

Hence we have the formula (2.6) . Similarly we have

$$(2.7)\quad \sum_{\substack{j_G(i)+1\\ i}} a_{+k}(y-k)v(y) - \sum_{\substack{j_L(i)-1\\ i}} a_{+k}(y)v(y) = \sum_{\Omega_h+\{j_L(i)-1\}} k a_{+k} v_y + \sum_{\Omega_h}\big(a_{+k}(y)-a_{+k}(y-k)\big)v$$

Now we define two functions ψ_1 and ψ_2 over a region $\Omega' \supset \Omega$ which are equal to cos(n,x)(*) and cos(n,y) respecti-

(*) cos(n,x) means the direction cosine between the outer normal and the x-axis.

-vely at a point on each side of $\Gamma_h + \Gamma_h'$, and which have bounded derivatives of the first order in Ω'. It is possible if the boundary Γ of Ω is smooth. The expression to be estimated can be written as follows ;

$$
\begin{aligned}
&\sum_{\Gamma_h} u^2 \Delta\Gamma + \sum_{\Gamma_h'} u^2 \Delta\Gamma \\
&= \sum_{\substack{i_G+1\\ j_G+1}} u^2 \left[\left(\frac{a_{-h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{-k}}{\Delta\Gamma}\right)^2\right]\Delta\Gamma + \sum_{\substack{i_G+1\\ j_L-1}} u^2 \left[\left(\frac{a_{-h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{+k}}{\Delta\Gamma}\right)^2\right]\Delta\Gamma \\
&+ \sum_{\substack{i_L-1\\ j_G+1}} u^2 \left[\left(\frac{a_{+h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{-k}}{\Delta\Gamma}\right)^2\right]\Delta\Gamma + {\sum_{\substack{i_L-1\\ j_L-1}}}' u^2 \left[\left(\frac{a_{+h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{+k}}{\Delta\Gamma}\right)^2\right]\Delta\Gamma \\
&+ {\sum_{\substack{i_L-1\\ j_L-1}}}'' u^2 \left[\left(\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{+k}}{\Delta\Gamma}\right)^2\right]\Delta\Gamma \\
&+ \sum_{\substack{i_G+1\\ \Delta}} u^2 \left[\left(\frac{a_{-h}}{\Delta\Gamma}\right)^2 + \left\{\frac{a_{+k}-a_{-k}}{\Delta\Gamma}\right\}^2_{x-h}\right]\Delta\Gamma + \sum_{\substack{i_L-1\\ \Delta}} u^2 \left[\left(\frac{a_{+h}}{\Delta\Gamma}\right)^2 + \left\{\frac{a_{+k}-a_{-k}}{\Delta\Gamma}\right\}^2_{x+h}\right]\Delta\Gamma \\
&+ \sum_{\substack{j_G+1\\ \Delta}} u^2 \left[\left(\frac{a_{-k}}{\Delta\Gamma}\right)^2 + \left\{\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right\}^2_{y-k}\right]\Delta\Gamma + \sum_{\substack{j_L-1\\ \Delta}} u^2 \left[\left(\frac{a_{+k}}{\Delta\Gamma}\right)^2 + \left\{\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right\}^2_{y+k}\right]\Delta\Gamma \\
&+ \sum_{\Gamma_h'} u^2 \left[\left(\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{+k}-a_{-k}}{\Delta\Gamma}\right)^2\right]\Delta\Gamma,
\end{aligned}
$$

where the right side is what was expanded in the case of Fig. 1. The symbols $\sum\limits_{\substack{i_G+1\\ j_G+1}}$ and ${\sum\limits_{\substack{i_L-1\\ j_L-1}}}'$ etc. mean the summation over the boundary mesh points of the first kind facing triangular meshes, and the symbol ${\sum\limits_{\substack{i_L-1\\ j_L-1}}}''$ denotes the quantity at the boundary mesh points of the first kind facing a quadrate mesh. Further $\sum\limits_{\substack{i_G+1\\ \Delta}}$ etc. denote the summation over the boundary mesh points of the second kind. In other cases we have only trivial changes.

Now the last equation becomes

$$\sum_{\Gamma_h} u^2 \Delta\Gamma + \sum_{\Gamma_h'} u^2 \Delta\Gamma$$

$$= \sum_{\substack{i_G(j)+1 \\ j}} a_{-h} u^2 \left(\frac{a_{-h}}{\Delta\Gamma}\right) - \sum_{\substack{i_L(j)-1 \\ j}}' a_{+h} u^2 \left(\frac{-a_{+h}}{\Delta\Gamma}\right) - \sum_{\substack{i_L-1 \\ j_L-1}}'' a_{+h} u^2 \left(-\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right)$$

$$+ \sum_{\substack{i_L-1 \\ j_L-1}}'' a_{-h} u^2 \left(-\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right) + \sum_{\substack{j_G(i)+1 \\ i}} a_{-k} u^2 \left(\frac{a_{-k}}{\Delta\Gamma}\right) - \sum_{\substack{j_L(i)-1 \\ i}} a_{+k} u^2 \left(\frac{-a_{+k}}{\Delta\Gamma}\right)$$

$$+ \sum_{\substack{i_G+1 \\ \Delta}} u^2 \left\{\frac{a_{+k}-a_{-k}}{\Delta\Gamma}\right\}^2_{x-h} \Delta\Gamma + \sum_{\substack{i_L-1 \\ \Delta}} u^2 \left\{\frac{a_{+k}-a_{-k}}{\Delta\Gamma}\right\}^2_{x+h} \Delta\Gamma +$$

$$+ \sum_{\substack{j_G+1 \\ \Delta}} u^2 \left\{\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right\}^2_{y-k} \Delta\Gamma + \sum_{\substack{j_L-1 \\ \Delta}} u^2 \left\{\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right\}^2_{y+k} \Delta\Gamma$$

$$+ \sum_{\Gamma_h'} u^2 \left[\left(\frac{a_{+h}-a_{-h}}{\Delta\Gamma}\right)^2 + \left(\frac{a_{+k}-a_{-k}}{\Delta\Gamma}\right)^2\right] \Delta\Gamma ,$$

where $\sum\limits_{\substack{i_L(j)-1 \\ j}}' = \sum\limits_{\substack{i_L-1 \\ j_G+1}} + \sum\limits_{\substack{i_L(j)-1 \\ \Delta}} + \sum\limits_{\substack{i_L-1 \\ j_L-1}}'$. By using the functions ψ_1 and ψ_2, we have

$$\sum_{\Gamma_h} u^2 \Delta\Gamma + \sum_{\Gamma_h'} u^2 \Delta\Gamma$$

$$= \sum_{\substack{i_G(j)+1 \\ j}} a_{-h} u^2 \psi_1 - \sum_{\substack{i_L(j)-1 \\ j}} a_{+h} u^2 \psi_1 + \sum_{i_L-1}'' a_{-h} u^2 \psi_1$$

$$+ \sum_{\substack{j_G(i)+1 \\ i}} a_{-k} u^2 \psi_2 - \sum_{\substack{j_L(i)-1 \\ i}} a_{+k} u^2 \psi_2$$

$$+ \sum_{\substack{i_G+1 \\ \Delta}} u^2 \psi_2^2 \Delta\Gamma + \sum_{\substack{i_L-1 \\ \Delta}} u^2 \psi_2^2 \Delta\Gamma$$

$$+ \sum_{\substack{j_G+1 \\ \Delta}} u^2 \psi_1^2 \Delta\Gamma + \sum_{\substack{j_L-1 \\ \Delta}} u^2 \psi_1^2 \Delta\Gamma + \sum_{\Gamma_h'} u^2 [\psi_1^2 + \psi_2^2] \Delta\Gamma$$

By applying (2.6) and (2.7), the last equation becomes

(2.8) $\sum\limits_{\Gamma_h} u^2 \Delta\Gamma + \sum\limits_{\Gamma_h'} u^2 \Delta\Gamma =$

$$= \sum_{\Omega_h+\{i_G(j)-1\}} h a_{+h}(\psi_1 u^2)_x + \sum_{\Omega_h} (a_{+h}(x)-a_{+h}(x-h))\psi_1 u^2$$

$$+ \sum_{\Omega_h+\{j_L(i)-1\}} k a_{+k}(\psi_2 u^2)_y + \sum_{\Omega_h} (a_{+k}(y)-a_{+k}(y-k))\psi_2 u^2$$

$$+ \sum_{\substack{i_L-1\\ j_L-1}}'' a_{-h}\psi_1 \left\{u^2 - 2kuu_{\bar{y}} + k^2 u_{\bar{y}}^2\right\}_{y+k}$$

$$+ \sum_{\substack{i_G+1\\ \Delta}} \psi_2^2 u^2 \Delta\Gamma + \sum_{\substack{i_L-1\\ \Delta}} \psi_2^2 u^2 \Delta\Gamma$$

$$+ \sum_{\substack{j_G+1\\ \Delta}} \psi_1^2 u^2 \Delta\Gamma + \sum_{\substack{j_L-1\\ \Delta}} \psi_1^2 u^2 \Delta\Gamma + \sum_{\Gamma_h'} [\psi_1^2 + \psi_2^2] u^2 \Delta\Gamma.$$

Here we note that because the equalities $a_{-h}(x)=a_{+h}(x-h)$ and $a_{-k}(y)=a_{+k}(y-k)$ hold at some interior mesh points of Ω_h, we have

$$\sum_{\Omega_h} (a_{+h}(x)-a_{+h}(x-h))\psi_1 u^2$$

$$= \Big(\sum_{\substack{j_G\\ (\Delta)}} + \sum_{\substack{j_L\\ (\Delta)}} + \sum_{\Gamma_h'} + \sum_{\Gamma_h''} \Big)(a_{+h}(x)-a_{+h}(x-h))\psi_1 u^2$$

and

$$\sum_{\Omega_h} (a_{+k}(y)-a_{+k}(y-k))\psi_2 u^2$$

$$= \Big(\sum_{\substack{i_G\\ (\Delta)}} + \sum_{\substack{i_L\\ (\Delta)}} + \sum_{\Gamma_h'} \Big)(a_{+k}(y)-a_{+k}(y-k))\psi_2 u^2,$$

where $\sum_{\substack{i_G\\ (\Delta)}}$ ($\sum_{\substack{i_L\\ (\Delta)}}$, $\sum_{\substack{j_G\\ (\Delta)}}$ and $\sum_{\substack{j_L\\ (\Delta)}}$) denotes the summation over the mesh points i_G (i_L, j_G and j_L) adjoining the boundary mesh point of the second kind. $\sum_{\Gamma_h''}$ denotes the summation over the mesh point adjoining two boundary mesh points of the first kind, one of which faces to a quadrate mesh and another to a triangular mesh. (See the point β in the Fig. 1). Now

$$\sum_{\substack{j_G\\(\Delta)}} (a_{+h}(x)-a_{+h}(x-h))\,\psi_1\, u^2 = -\sum_{\substack{j_G\\(\Delta)}} \psi_1(y+k)\psi_1(y)u^2\Delta\Gamma$$

$$= -\sum_{\substack{j_G+1\\\Delta}} \psi_1^2\left[u^2-2kuu_{\bar{y}}+k^2u_{\bar{y}}^2\right]\Delta\Gamma + \sum_{\substack{j_G\\(\Delta)}} k\,\psi_1(y+k)\,\psi_{1y}(y)u^2\Delta\Gamma \;,$$

$$\sum_{\substack{j_L\\(\Delta)}} (a_{+h}(x)-a_{+h}(x-h))\,\psi_1\, u^2$$

$$= -\sum_{\substack{j_L-1\\\Delta}} \psi_1^2\left[u^2+2kuu_y+k^2u_y^2\right]\Delta\Gamma - \sum_{\substack{j_L\\(\Delta)}} k\,\psi_1(y-k)\,\psi_{1\bar{y}}(y)u^2\Delta\Gamma \;,$$

$$\sum_{\substack{i_G\\(\Delta)}} (a_{+k}(y)-a_{+k}(y-k))\,\psi_2\, u^2$$

$$= -\sum_{\substack{i_G+1\\\Delta}} \psi_2^2\left[u^2-2huu_{\bar{x}}+h^2u_{\bar{x}}^2\right]\Delta\Gamma + \sum_{\substack{i_G\\(\Delta)}} h\,\psi_2(x+h)\,\psi_{2x}(x)u^2\Delta\Gamma$$

and

$$\sum_{\substack{i_L\\(\Delta)}} (a_{+k}(y)-a_{+k}(y-k))\,\psi_2\, u^2$$

$$= -\sum_{\substack{i_L-1\\\Delta}} \psi_2^2\left[u^2+2huu_x+h^2u_x^2\right]\Delta\Gamma - \sum_{\substack{i_L\\(\Delta)}} h\,\psi_2(x-h)\,\psi_{2\bar{x}}u^2\Delta\Gamma \;.$$

We note also that

$$\sum_{\Gamma_h'} (a_{+h}(x)-a_{+h}(x-h))\,\psi_1\, u^2 + \sum_{\Gamma_h'} (a_{+k}(y)-a_{+k}(y-k))\,\psi_2\, u^2$$

$$= -\sum_{\Gamma_h'} (\psi_1^2+\psi_2^2)u^2\,\Delta\Gamma$$

and

$$\sum_{\Gamma_h''} (a_{+h}(x)-a_{+h}(x-h))\,\psi_1\, u^2 = \sum_{\beta} (a_{+h}(x)-a_{+h}(x-h))\,\psi_1\, u^2$$

$$= -\sum_{\substack{i_L-1\\j_L-1}}'' a_{-h}\psi_1 u(y+k)^2 - \sum_{\substack{i_L-1\\j_L-1}}'' a_{-h}k\,\psi_{1y}u(y+k)^2 \;.$$

Therefore from (2.8), we have

$$(2.9)\quad \sum_{\Gamma_h} u^2\Delta\Gamma + \sum_{\Gamma_h'} u^2\Delta\Gamma$$

$$= \sum_{\Omega_h+\{i_L(j)-1\}} ha_{+h}(\psi_1 u^2)_x + \sum_{\Omega_h+\{j_L(i)-1\}} ka_{+k}(\psi_2 u^2)_y$$

$$-\sum_{\substack{i_G+1\\ \Delta}} \psi_2^2\left[-2huu_{\bar x}+h^2u_{\bar x}^2\right]\Delta\Gamma + \sum_{\substack{i_G\\ (\Delta)}} h\,\psi_2(x+h)\,\psi_{2x}(x)u^2\Delta\Gamma \; -$$

$$-\sum_{\substack{i_L-1\\ \Delta}} \psi_2^2\left[\;2huu_x+h^2u_x^2\right]\Delta\Gamma - \sum_{\substack{i_L\\ (\Delta)}} h\,\psi_2(x-h)\,\psi_{2\bar x}u^2\Delta\Gamma\cdot$$

$$-\sum_{\substack{j_G+1\\ \Delta}} \psi_1^2\left[-2kuu_{\bar y}+k^2u_{\bar y}^2\right]\Delta\Gamma + \sum_{\substack{j_G\\ (\Delta)}} k\,\psi_1(y+k)\,\psi_{1y}(y)u^2\Delta\Gamma$$

$$-\sum_{\substack{j_L-1\\ \Delta}} \psi_1^2\left[\;2kuu_y+k^2u_y^2\right]\Delta\Gamma - \sum_{\substack{j_L\\ (\Delta)}} k\,\psi_1(y-k)\,\psi_{1\bar y}(y)u^2\Delta\Gamma$$

$$+\sum_{\substack{i_L-1\\ j_L-1}}{}'' a_{-h}\psi_1\left\{-2kuu_{\bar y}+k^2u_{\bar y}^2\right\}_{y+k} - a_{-h}k\,\psi_{1y}u(y+k)^2.$$

Applying the formulae

$$(\psi_1 u^2)_x = \psi_{1x}u^2 + \psi_1(x+h)u_xu + \{\psi_1 uu_{\bar x}\}_{x+h}\;,$$

$$(\psi_2 u^2)_y = \psi_{2y}u^2 + \psi_2(y+k)u_yu + \{\psi_2 uu_{\bar y}\}_{y+k}$$

and using the Schwarz's inequality in the right side of (2.9), we have the estimate

$$\sum_{\Gamma_h} u^2\Delta\Gamma + \sum_{\Gamma_h'} u^2\Delta\Gamma$$

$$\leqq \varepsilon\sum_{\Omega_h}\left[h(a_{+h}u_x^2+a_{-h}u_{\bar x}^2)+k(a_{+k}u_y^2+a_{-k}u_{\bar y}^2)\right] + C(\varepsilon)\sum_{\Omega_h} S_h u^2$$

$$+\;O(h)\sum_{\Gamma_h'} u^2\Delta\Gamma\;,$$

where $C(\varepsilon)$ depends on ε and the bound of $\psi_1, \psi_2, \frac{\partial \psi_1}{\partial x}, \cdots, \frac{\partial \psi_2}{\partial y}$ etc.
From the last equation we get Lemma 1 for small h.

<u>Lemma 2</u> For any u defined on $\Omega_h \times [s\Delta t;\ s=0,1,\cdots,p,\ p\Delta t \leq T]$, we have the inequalities

$$(2.10) \quad \sum_{\Omega_h} S_h u^2 (p\Delta t) \leq 2T \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h u_{\bar{t}}^2 (s\Delta t) + 2 \sum_{\Omega_h} S_h u(o)^2$$

and

$$(2.11) \quad \sum_{s=0}^{p} \Delta t \sum_{\Omega_h} S_h u^2 (s\Delta t) \leq 2T^2 \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h u_{\bar{t}}^2 (s\Delta t) + 2T \sum_{\Omega_h} S_h u(o)^2 .$$

The proof of Lemma 2 is easy. In fact

$$u(p\Delta t) = \sum_{s=1}^{p} \Delta t u_{\bar{t}}(s\Delta t) + u(o),$$

$$u^2 (p\Delta t) \leq 2\left(p\Delta t \sum_{s=1}^{p} \Delta t u_{\bar{t}}^2 (s\Delta t) + u(o)^2\right).$$

Hence we have the desired inequalities (2.10) and (2.11).

Now we return to the inequality (2.5). At first we note that by the boundary conditions

$$\sum_{\Gamma_h} u_{\bar{t}}^2 \Delta\Gamma \leq \frac{C_2}{\sqrt{hk}} \sum_{\Omega_h} S_h u_{\bar{t}}^2 + C_3 h^2 \sum_{\Gamma_h} g_{\bar{t}}^2 \Delta\Gamma ,$$

Using Lemma 1, Lemma 2 and the last equation, we have the following estimtes about the term of the summation $\sum_{\Gamma_h} + \sum_{\Gamma_h'}$ on the right side of (2.5),

$$\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma\Big[(1+\tfrac{3}{2}|\delta|)u^2+\tfrac{\Delta t^2}{2}(1+|\delta|)u_{\bar t}^2\Big]_{p\Delta t}$$

$$\leq (1+\tfrac{3}{2}|\delta|)\varepsilon\sum_{\Omega_h}\Big[h(a_{+h}u_x^2+a_{-h}u_{\bar x}^2)+k(a_{+k}u_y^2+a_{-k}u_{\bar y}^2)\Big]_{p\Delta t}$$

$$+(1+\tfrac{3}{2}|\delta|)C(\varepsilon)\sum_{\Omega_h}S_h u(p\Delta t)^2$$

$$+\frac{C_2\Delta t^2}{2\sqrt{hk}}(1+|\delta|)\sum_{\Omega_h}S_h u_{\bar t}\ (p\Delta t)^2+\frac{C_3\Delta t^2 h^2}{2}(1+|\delta|)\sum_{\Gamma_h}g_{\bar t}^2\ (p\Delta t)\Delta\Gamma$$

$$\leq \varepsilon'\sum_{\Omega_h}\Big[h(a_{+h}u_x^2+a_{-h}u_{\bar x}^2)+k(a_{+k}u_y^2+a_{-k}u_{\bar y}^2)\Big]_{p\Delta t}$$

$$+C_4(\varepsilon')\sum_{s=1}^{p}\Delta t\sum_{\Omega_h}S_h u_{\bar t}\ (s\Delta t)^2$$

$$+C_5\frac{\Delta t^2}{\sqrt{hk}}\sum_{\Omega_h}S_h u_{\bar t}\ (p\Delta t)^2+C_4'(\varepsilon')\sum_{\Omega_h}S_h u(o)^2$$

$$+C_3'\Delta t^2 h^2\sum_{\Gamma_h}g_{\bar t}\ (p\Delta t)^2\Delta\Gamma,$$

where $\varepsilon'=(1+\frac{3}{2}|\delta|)\varepsilon$, $C_4(\varepsilon')=2T(1+\frac{3}{2}|\delta|)C(\varepsilon)$, $C_5=\frac{C}{2}(1+|\delta|)$, $C_4'(\varepsilon')=2(1+\frac{3}{2}|\delta|)C(\varepsilon)$ and $C_3'=\frac{3}{2}(1+|\delta|)$.

And

$$\sum_{s=0}^{p-1}\Delta t\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)u^2(s\Delta t)^2\Delta\Gamma$$

$$\leq \varepsilon\sum_{s=0}^{p-1}\Delta t\sum_{\Omega_h}\Big[h(a_{+h}u_x^2+a_{-h}u_{\bar x}^2)+k(a_{+k}u_y^2+a_{-k}u_{\bar y}^2)\Big]_{s\Delta t}$$

$$+C(\varepsilon)\sum_{s=0}^{p-1}\Delta t\sum_{\Omega_h}S_h u_{\bar t}(s\Delta t)^2$$

$$\leq \varepsilon\sum_{s=0}^{p-1}\Delta t\sum_{\Omega_h}\Big[h(a_{+h}u_x^2+a_{-h}u_{\bar x}^2)+k(a_{+k}u_y^2+a_{-k}u_{\bar y}^2)\Big]_{s\Delta t}$$

$$+C_6(\varepsilon)\sum_{s=0}^{p-1}\Delta t\sum_{\Omega_h}S_h u_{\bar t}\ (s\Delta t)^2+C_6(\varepsilon)'\sum_{\Omega_h}S_h u(o)^2$$

where $C_6(\varepsilon)=2T^2C(\varepsilon)$ and $C_6(\varepsilon)'=2TC(\varepsilon)$. By applying these results in(2.5), the following inequality holds:

$$(2.12)\left\{ \sum_{\substack{i_G(j)\\ j}}\left[\frac{S_h'}{2}-\Delta t\left(\frac{a_{+h}}{4}+\frac{a_{-h}}{8}\right)\right]u_{\bar t}^2+\sum_{\substack{i_L(j)\\ j}}\left[\frac{S_h'}{2}-\Delta t\left(\frac{a_{-h}}{4}+\frac{a_{+h}}{8}\right)\right]u_{\bar t}^2\right.$$

$$+\sum_{\substack{j_G(i)\\ i}}\left[\frac{S_h'}{2}-\Delta t\left(\frac{a_{+k}}{4}+\frac{a_{-k}}{8}\right)\right]u_{\bar t}^2+\sum_{\substack{j_L(i)\\ i}}\left[\frac{S_h'}{2}-\Delta t\left(\frac{a_{-k}}{4}+\frac{a_{+k}}{8}\right)\right]u_{\bar t}^2$$

$$+\sum_{\substack{i\neq i_G(j),\, i_L(j)\\ j}}\left[\frac{S_h'}{2}-\Delta t\left(\frac{a_{+h}+a_{-h}}{4}\right)\right]u_{\bar t}^2+\sum_{\substack{j\neq j_G(i),\, j_L(i)\\ i}}\left[\frac{S_h'}{2}-\Delta t\left(\frac{a_{+k}+a_{-k}}{4}\right)\right]u_{\bar t}^2$$

$$+\sum_{\substack{i_G(j)\\ j}}ha_{+h}\left(1-\frac{2\Delta t}{h}-\varepsilon'\right)u_x^2+\frac{1}{2}\sum_{\substack{i\neq i_G(j)\\ j}}ha_{+h}\left(1-\frac{2\Delta t}{h}-2\varepsilon'\right)u_x^2$$

$$+\sum_{\substack{i_L(j)\\ j}}ha_{-h}\left(1-\frac{2\Delta t}{h}-\varepsilon'\right)u_{\bar x}^2+\frac{1}{2}\sum_{\substack{i\neq i_L(j)\\ j}}ha_{-h}\left(1-\frac{2\Delta t}{h}-2\varepsilon'\right)u_{\bar x}^2$$

$$+\sum_{\substack{j_G(i)\\ i}}ka_{+k}\left(1-\frac{2\Delta t}{k}-\varepsilon'\right)u_y^2+\frac{1}{2}\sum_{\substack{j\neq j_G(i)\\ i}}ka_{+k}\left(1-\frac{2\Delta t}{k}-2\varepsilon'\right)u_y^2$$

$$\left.+\sum_{\substack{j_L(i)\\ i}}ka_{-k}\left(1-\frac{2\Delta t}{k}-\varepsilon'\right)u_{\bar y}^2+\frac{1}{2}\sum_{\substack{j\neq j_L(i)\\ i}}ka_{-k}\left(1-\frac{2\Delta t}{k}-2\varepsilon'\right)u_{\bar y}^2\right\}_{p\Delta t}$$

$$\leqq C_7(\varepsilon)\sum_{s=1}^{p}\Delta t\sum_{\Omega_h}S_h u_{\bar t}(s\Delta t)^2$$

$$+\varepsilon\sum_{s=1}^{p}\Delta t\sum_{\Omega_h}\left[h(a_{+h}u_x^2+a_{-h}u_{\bar x}^2)+k(a_{+k}u_y^2+a_{-k}u_{\bar y}^2)\right]_{s\Delta t}$$

$$+F(p)+G(p)+\Phi,$$

where $S_h'=S_h\left(1-C_1\Delta t-C_5\frac{\Delta t^2}{\sqrt{hk}}\right)$, $C_7(\varepsilon)=1+C_4(\varepsilon')+C_6(\varepsilon)$,

$$F(p)=\sum_{s=1}^{p-1}\Delta t\sum_{\Omega_h}S_h f^2,$$

$$G(p)=\left(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\right)\Delta\Gamma\left[2\bar g((p-1)\Delta t)^2+\Delta t^2\left(1+\frac{C_2\sqrt{hk}}{2}\right)g_{\bar t}(p\Delta t)^2\right]$$

$$+\sum_{s=0}^{p-1}\Delta t\left(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\right)\Delta\Gamma\,\bar g_t^2+\frac{1}{2}\left(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\right)\Delta\Gamma\left[\bar g(0)^2+(1+\Delta t)\bar g(\Delta t)^2\right]$$

$$+C_3'\Delta t^2h^2\sum_{\Gamma_h}g_{\bar t}(p\Delta t)^2\Delta\Gamma$$

and

$$\Phi = \frac{1}{2}\Big\{ \Big(\sum_{\substack{i_R(j)\\ j}} + \sum_{\Omega_h}\Big) ha_{+h}\Big[\Big(1+\frac{\Delta t}{2}\Big)\varphi_x^2 + \frac{\Delta t}{2}\psi_x^2\Big]$$

$$+ \Big(\sum_{\substack{i_L(j)\\ j}} + \sum_{\Omega_h}\Big) ha_{-h}\Big[\Big(1+\frac{\Delta t}{2}\Big)\varphi_{\bar{x}}^2 + \frac{\Delta t}{2}\psi_{\bar{x}}^2\Big]$$

$$+ \Big(\sum_{\substack{j_R(i)\\ i}} + \sum_{\Omega_h}\Big) ka_{+k}\Big[\Big(1+\frac{\Delta t}{2}\Big)\varphi_y^2 + \frac{\Delta t}{2}\psi_y^2\Big]$$

$$+ \Big(\sum_{\substack{j_L(i)\\ i}} + \sum_{\Omega_h}\Big) ka_{-k}\Big[\Big(1+\frac{\Delta t}{2}\Big)\varphi_{\bar{y}}^2 + \frac{\Delta t}{2}\psi_{\bar{y}}^2\Big]\Big\}$$

$$+ \sum_{\Omega_h} S_h\psi^2 + (C_4(\varepsilon') + C_6(\varepsilon)') \sum_{\Omega_h} S_h\varphi^2$$

$$+ \frac{1}{2}\Big(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\Big)\Delta\Gamma\big[\varphi^2 + \Delta t\psi^2\big] .$$

Now we choose Δt, h and k as follows:

$$1 - \frac{2\Delta t}{h} \geqq 2\alpha > 0, \quad 1 - \frac{2\Delta t}{k} \geqq 2\alpha > 0 ,$$

where α is a small constant. Then for sufficiently small Δt

$$\frac{S_h'}{2} - \Delta t\Big(\frac{a_{+h}}{4} + \frac{a_{-h}}{8}\Big) = \frac{S_h}{2} - \frac{\Delta t(a_{+h}+a_{-h})}{4} + \Delta t\Big(\frac{a_{-h}}{8} - \frac{C_1 S_h}{2} - \frac{C_5 S_h \Delta t}{2\sqrt{hk}}\Big)$$

$$\geqq \frac{S_h}{2}\Big(1 - \frac{2\Delta t}{h}\,\frac{h}{a_{+k}+a_{-k}}\Big) \geqq \alpha S_h , \qquad \text{etc.}$$

and

$$\frac{S_h'}{2} - \Delta t\Big(\frac{a_{+h}+a_{-h}}{4}\Big) \geqq \frac{S_h}{2}\Big(1 - \frac{2\Delta t}{h}\,\frac{h}{a_{+k}+a_{-k}}\Big) - \frac{S_h}{2}\Big(C_1\Delta t + C_5\frac{\Delta t^2}{\sqrt{hk}}\Big)$$

$$\geqq \frac{\alpha}{2} S_h \qquad \text{etc.}$$

Further we can take $\varepsilon'(\varepsilon)$ so small that

$$ha_{+h}\Big(1 - \frac{2\Delta t}{h} - \varepsilon'\Big) \geqq \alpha h a_{+h} \qquad \text{etc.}$$

Substituting these inequalities into (2.12) we have,

$$\alpha \sum_{\Omega_h} S_h u_{\bar{t}}(p\Delta t)^2 + \frac{\alpha}{2} \sum_{\Omega_h} \left[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2) + k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2) \right]_{p\Delta t}$$

$$\leq C_7(\varepsilon) \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h u_{\bar{t}}\ (s\Delta t)^2$$

$$+ \varepsilon \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} \left[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2) + k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2) \right]_{s\Delta t}$$

$$+ F(p) + G(p) + \Phi .$$

We set

$$\sum_{s=1}^{p} \Delta t \sum_{\Omega_h} \left[S_h u_{\bar{t}}^2 + \frac{h}{2} (a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2) + \frac{k}{2} (a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2) \right]_{s\Delta t}$$

$$= S(p)$$

Then the last inequality can be written in the form

$$\text{(2.13)} \qquad \frac{S(p) - S(p-1)}{\Delta t} \leq C_8 S(p) + \Psi(p),$$

where $C = \max\left(\frac{C_7(\varepsilon)}{\alpha}, \frac{2\varepsilon}{\alpha}\right)$, $\Psi(p) = \frac{1}{\alpha}(F(p) + G(p) + \Phi)$.

Hence we find that

$$S(p) \leq \frac{1}{1-C_8\Delta t} S(p-1) + \frac{\Delta t}{1-C_8\Delta t} \Psi(p) .$$

We choose Δt so small that $1-C_8\Delta t \geq \frac{1}{2}$ and we put $\frac{1}{1-C_8\Delta t} = R$. Then

$$S(p) \leq RS(p-1) + R\Delta t \Psi(p) .$$

We apply the last inequality successively, and then

$$S(p) \leqq R^{p-1} S(1) + R\Delta t \sum_{s=2}^{p} R^{p-1} \Psi(s).$$

However

$$R^p = \left(1 + \frac{C_8 \Delta t}{1 - C_8 \Delta t}\right)^p \leqq e^{\frac{C_8 p \Delta t}{1 - C_8 \Delta t}} \leqq e^{2C_8 T},$$

since

$$C_8 \Delta t \leqq \frac{1}{2}, \quad p\Delta t \leqq T.$$

Therefore we get

$$S(p) \leqq e^{2C_8 T}\left[S(1) + T\Psi(p)\right]$$

Finally we estimate $S(1)$;

$$S(1) = \sum_{\Omega_h}\Big[S_h \psi^2 + \frac{h}{2} a_{+h}(\varphi_x + \Delta t \psi_x)^2 + \frac{h}{2} a_{-h}(\varphi_{\bar{x}} + \Delta t \psi_{\bar{x}})^2 +$$

$$+ \frac{k}{2} a_{+k}(\varphi_y + \Delta t \psi_y)^2 + \frac{k}{2} a_{-k}(\varphi_{\bar{y}} + \Delta t \psi_{\bar{y}})^2\Big]$$

$$\leqq \sum_{\Omega_h}\Big[S_h \psi^2 + \frac{3ha_{+h}}{2}\Big\{\varphi_x^2 + \left(\frac{\Delta t}{h}\right)^2 \psi(x+h)^2 + \left(\frac{\Delta t}{h}\right)^2 \psi(x)^2\Big\}$$

$$+ \frac{3ha_{-h}}{2}\Big\{\varphi_{\bar{x}}^2 + \left(\frac{\Delta t}{h}\right)^2 \psi(x-h)^2 + \left(\frac{\Delta t}{h}\right)^2 \psi(x)^2\Big\}$$

$$+ \frac{3ka_{+k}}{2}\Big\{\varphi_y^2 + \left(\frac{\Delta t}{k}\right)^2 \psi(y+k)^2 + \left(\frac{\Delta t}{k}\right)^2 \psi(y)^2\Big\}$$

$$+ \frac{3ka_{-k}}{2}\Big\{\varphi_{\bar{y}}^2 + \left(\frac{\Delta t}{k}\right)^2 \psi(y-k)^2 + \left(\frac{\Delta t}{k}\right)^2 \psi(y)^2\Big\}\Big]$$

$$\leqq C_9 \sum_{\Omega_h} S_h(\psi^2 + \varphi_x^2 + \varphi_{\bar{x}}^2 + \varphi_y^2 + \varphi_{\bar{y}}^2),$$

where we used the fact that the functions φ and ψ are smooth in $\Omega' \supset \Omega$. Consequently we have

$$S(p) \leqq C_{10}(T)\, \Phi(p),$$

i.e.,
$$\sum_{s=1}^{p} \Delta t \sum_{\Omega_h} \left[S_h u_{\bar{t}}^2 + \frac{h}{2}\,(a_{+h} u_x^2 + a_{-h} u_{\bar{x}}^2) + \frac{k}{2}\,(a_{+k} u_y^2 + a_{-k} u_{\bar{y}}^2) \right]_{s\Delta t}$$
$$\leqq C_{10}(T)\, \Phi(p).$$

From the last inequality and (2.13) we have also a strong inequality

$$\sum_{\Omega_h} \left[S_h u_{\bar{t}}^2 + \frac{h}{2}\,(a_{+h} u_x^2 + a_{-h} u_{\bar{x}}^2) + \frac{k}{2}\,(a_{+k} u_y^2 + a_{-k} u_{\bar{y}}^2) \right]_{p\Delta t}$$
$$\leqq C_{11}(T)\, \Phi(p). \tag{2.14}$$

Thus we have

<u>Theorem 1</u> (Stability) Suppose that Δt, h and k satisfy the condition

$$1 - \frac{2\Delta t}{h} \geqq \alpha_1 > 0\,, \qquad 1 - \frac{2\Delta t}{k} \geqq \alpha_1 > 0\,, \tag{2.15}$$

where α_1 is an arbitrarily small constant. Then the solution u of the difference problem (1.9), (1.7), (1.10), (1.11) and (1.12) satisfies an energy inequality (2.14) for sufficiently small Δt and $p\Delta t \leqq T$.

Immediately from the last theorem we have a convergence theorm. In fact weexpressthe solution of the differential problem (1.1), (1.2) and (1.3) by vandthe corresponding solution of the difference problem (1.9), (1.7), (1.10), (1.11), and (1.12) by u. And we put w=v-u. Let's consider again the hatched mesh in Fig.1. Then we have

$$S_h v_{t\bar{t}} = \iint \frac{\partial^2 v}{\partial t^2}\,dxdy + M_1 h S_h\,,$$

$$(\bar{g}+\delta\{v\}_{x+h}) \quad -a_{-h}v_{\bar{x}}+a_{+k}v_y-a_{-k}v_{\bar{y}} = \oint \frac{\partial v}{\partial n}\,d\Gamma + \sum_i hM_{2i}\,\Delta\Gamma_i\,,$$

$$S_h\bar{f} = \iint f\,dxdy\,,$$

under the condition that $\frac{\partial^3 v}{\partial t^3}$, $\frac{\partial^3 v}{\partial x\partial t^2}$, $\frac{\partial^3 v}{\partial y\partial t^2}$, $\frac{\partial^2 v}{\partial x^2}$, $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial^2 v}{\partial x\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous, where M_1 depends on the magnitude of $\frac{\partial^3 v}{\partial t^3}$, $\frac{\partial^3 v}{\partial x\partial t^2}$ and $\frac{\partial^3 v}{\partial y\partial t^2}$, M_{2i} depend on that of $\frac{\partial^2 v}{\partial x^2}$, $\frac{\partial^2 v}{\partial y^2}$, $\frac{\partial^2 v}{\partial x\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, and $\Delta\Gamma_i$ are the lengths of sides of the trapezoid.

Hence we have

$$S_h v_{t\bar{t}} - \left[(\bar{g} + \delta\{v\}_{x+h}) - a_{-h}v_{\bar{x}} + a_{+k}v_y - a_{-k}v_{\bar{y}}\right] - S_h\bar{f}$$

$$= \iint \frac{\partial^2 v}{\partial t^2}dxdy - \oint \frac{\partial v}{\partial n}d\Gamma - \iint f\,dxdy + M_1 h S_h - \sum_i hM_{2i}\,\Delta\Gamma_i$$

$$= \iint\left[\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} - f\right]dxdy + M_1 h S_h - \sum_i hM_{2i}\,\Delta\Gamma_i$$

$$= M_1 h S_h - \sum_i hM_{2i}\,\Delta\Gamma_i\,,$$

since v is the solution of (1.1). Therefore on the concerned mesh point w satisfies the equation

$$S_h w_{t\bar{t}} = \delta\{w\}_{x+h} - a_{-h}w_{\bar{x}} + a_{+k}w_y - a_{-k}w_{\bar{y}}$$

$$+ M_1 h S_h - \sum_i h M_{2i} \Delta\Gamma_i ,$$

which can be written in the form

$$S_h w_{t\bar{t}} = (g_1 + \delta\{w\}_{x+h}) - a_{-h}w_{\bar{x}} + a_{+k}w_y - a_{-k}w_{\bar{y}} + S_h f_1 ,$$

where $|g_1| \leq \text{const.} h$ and $f_1 = M_1 h$. Analogous difference equations are satisfied by w on the mesh points adjoining to the boundary mesh point of second kind.

In the same way we have the equation satisfied by w at the interior mesh point;

$$S_h w_{t\bar{t}} = a_{+h}w_x - a_{-h}w_{\bar{x}} + a_{+k}w_y - a_{-k}w_{\bar{y}} + S_h f_1 , \quad f_1 = M_3 h ,$$

where M_3 depends on the magnitude of the third derivatives of v with respect to t, x and y.

Therefore we have the difference equations over the region Ω_h in the same form as in (1.9);

$$S_h w_{t\bar{t}} = a_{+h}w_x - a_{-h}w_{\bar{x}} + a_{+k}w_y - a_{-k}w_{\bar{y}} + \delta_h(g_1 + \delta w)\Delta\Gamma + S_h f_1$$

under the boundary conditions in the form as in (1.7);

$$a_{-h}w_{\bar{x}} = (g_1 + \delta w)\Delta\Gamma , \quad a_{-h} = k \qquad \text{on a right boundary mesh point,}$$

$$a_{+h}w_x = -(g_1 + \delta w)\Delta\Gamma , \quad a_{+h} = k \qquad \text{on a left boundary mesh point,}$$

$$a_{-k} w_{\bar{y}} = (g_1 + \delta w) \quad , \quad a_{-k} = h \qquad \text{on an upper boundary mesh point,}$$

$$a_{+k} w_y = -(g_1 + \delta w) \quad , \quad a_{+k} = h \qquad \text{on a lower boundary mesh point.}$$

At the boundary point of first kind, as we know, the formulae (1.10) and (1.11) approximate formally the original boundary condition $\frac{\partial u}{\partial n} - \delta u = g$ with the error of order $O(h)$. This means

$$a_{-h} w_{\bar{x}} + a_{-k} w_{\bar{y}} = (g_1 + \delta w) \qquad \text{at a right upper mesh point,}$$

$$a_{-h} w_{\bar{x}} - a_{+k} w_y = (g_1 + \delta w) \qquad \text{at a right lower mesh point,}$$

$$-a_{+h} w_x + a_{-k} w_{\bar{y}} = (g_1 + \delta w) \qquad \text{at a left upper mesh point,}$$

$$-a_{+h} w_x - a_{+k} w_y = (g_1 + \delta w) \qquad \text{at a left lower mesh point,}$$

and

(in place of (1.11))

$$-a_{+h} w_x + a_{-h} \{w_{\bar{x}}\}_{y+k} - a_{+k} w_y = (g_1 + \delta w) \Delta\Gamma \ .$$

In all the above formulae about w we can recognize that $g_1 = O(h)$, $f_1 = O(h)$. So that we find that w satisfies the difference problem with $\varphi = o$, $\phi = O(\Delta t)$, $g = O(h)$ and $f = O(h) + O(\Delta t^2)$. Therefore we have from (2.14), using Lemma 1,

(2.16) $\quad \|w\|_h = \|v-u\|_h = O(h) \quad$ when $\quad h \to o$

uniformly for $p\Delta t \leqslant T$, where

$$\|w\|_h = \sum_{\Omega_h}\left[S_h w_{\bar{t}}^2 + \frac{h}{2}\,(a_{+h}w_x^2 + a_{-h}w_{\bar{x}}^2) + \frac{k}{2}\,(a_{+k}w_y^2 + a_{-k}w_{\bar{y}}^2) + S_h w^2\right]$$

and we supposed that a solution of the differential problem has continuous third derivatives in $\bar{\Omega}\times[o,T]$. Hence we have

Theorem 2 (Convergence) Suppose that the boundary Γ of Ω is sufficiently smooth and a solution of the differential problem (1.1), (1.2) and (1.3) has the regularity just stated. When Δt, h and k tend to zero under the condition (2.15), a solution of our difference problem converges to the corresponding solution of the differential problem in the sense of (2.16).

§3 A parabolic problem

We can construct an analogous difference scheme for a mixed initial-boundary problem for an equation of heat conduction

(3.1) $\qquad \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f \qquad$ in $Q(T)$,

(3.2) $\qquad \frac{\partial u}{\partial n} - \delta u = g \quad , \qquad$ on $\Gamma \times [0, T]$, (*)

(3.3) $\qquad u(0, x, y) = \varphi(x, y)$.

Here the notations are the same as in §1. As an approximation of the differential equation (3.1) we set an implicit difference equation analogous to (1.9),

(3.4) $\qquad S_h u_{\bar{t}} = a_{+h} u_x - a_{-h} u_{\bar{x}} + a_{+k} u_y - a_{-k} u_{\bar{y}} + \delta_h (\bar{g} + \delta u) \Delta\Gamma + S_h \bar{f}$.

As boundary conditions we use the formulae (1.7), (1.10) and (1.11). As initial conditions we use

(3.5) $\qquad u(0) = \varphi$.

In the same way as in §2 we can prove a stability theorem and a convergence theorem. For the proof we multiply the equation (3.4) by $\Delta t u$ and sum all over

$$\Omega_h \times \left[t = s\Delta t;\ s = 1, 2, \cdots, p \right], \quad \text{then}$$

(*) As in §1, we assume here and in §4 that at every point $P \in \Gamma$ there is a circle S such that $\bar{S} \cap \bar{\Omega} = P$.

$$(3.6)\qquad \sum_{s=1}^{p}\Delta t\sum_{\Omega_h} S_h u_{\bar t}u$$

$$= \frac{1}{2}\sum_{s=1}^{p}\Delta t\sum_{\Omega_h}\left[h(a_{+h}u_x)_{\bar x}u + h(a_{-h}u_{\bar x})_x u + k(a_{+k}u_y)_{\bar y}u + k(a_{-k}u_{\bar y})_y u\right]$$

$$+ \sum_{s=1}^{p}\Delta t\sum_{\Omega_h}\left[\delta_h(\bar g+\delta u)u\Delta\Gamma + S_h\bar f u\right].$$

However we have

$$\sum_{s=1}^{p}\Delta t u_{\bar t}u = \frac{1}{2}\sum_{s=1}^{p}\Delta t^2 u_{\bar t}^2 + \frac{1}{2}u(p\Delta t)^2 - \frac{1}{2}u(o)^2 .$$

and

$$\sum_{\Omega_h} h(a_{+h}u_x)_{\bar x}u = \sum_{\Omega_h} h\left[(a_{+h}u_x u)_{\bar x} - \left\{a_{+h}u_x^2\right\}_{x-h}\right]$$

$$= \sum_{\substack{i_G(j)\\ j}} a_{+h}u_x u - \sum_{\substack{i_L(j)-1\\ j}} a_{+h}u_x u - \sum_{\Omega_h} h a_{-h}u_{\bar x}^2$$

$$= \sum_{\substack{i_G(j)+1\\ j}} a_{-h}u_{\bar x}u - \sum_{\substack{i_L(j)-1\\ j}} a_{+h}u_x u - \sum_{\substack{i_G(j)\\ j}} h a_{+h}u_x^2 - \sum_{\Omega_h} h a_{-h}u_{\bar x}^2 ,$$

$$\sum_{\Omega_h} h(a_{-h}u_x)_{\bar x}u$$

$$= \sum_{\substack{i_G(j)+1\\ j}} a_{-h}u_{\bar x}u - \sum_{\substack{i_L(j)-1\\ j}} a_{+h}u_x u - \sum_{\substack{i_L(j)\\ j}} h a_{-h}u_{\bar x}^2 - \sum_{\Omega_h} h a_{+h}u_x^2 ,$$

$$\sum_{\Omega_h} k(a_{+k}u_y)_{\bar y}u$$

$$= \sum_{\substack{j_G(i)+1\\ i}} a_{-k}u_{\bar y}u - \sum_{\substack{j_L(i)-1\\ i}} a_{+k}u_y u - \sum_{\substack{j_G(i)\\ i}} k a_{+k}u_y^2 - \sum_{\Omega_h} k a_{-k}u_{\bar y}^2 ,$$

$$\sum_{\Omega_h} k(a_{-k}u_{\bar{y}})_y u$$

$$= \sum_{\substack{j_G(i)+1 \\ i}} a_{-k}u_{\bar{y}}u - \sum_{\substack{j_L(i)-1 \\ i}} a_{+k}u_y u - \sum_{\substack{j_L(i) \\ i}} ka_{-k}u_{\bar{y}}^2 - \sum_{\Omega_h} ka_{+k}u_y^2 .$$

Therefore the equation (3.6) can be written in the form

$$\frac{1}{2}\sum_{\Omega_h} S_h\left\{u(p\Delta t)^2 - u(o)^2\right\} + \frac{\Delta t}{2}\sum_{s=1}^{p}\Delta t\sum_{\Omega_h} S_h u_{\bar{t}}^2$$

$$= \sum_{s=1}^{p}\Delta t\Big[\sum_{\substack{i_G(j)+1 \\ j}} a_{-h}u_{\bar{x}}u - \sum_{\substack{i_L(j)-1 \\ j}} a_{+h}u_x u + \sum_{\substack{j_G(i)+1 \\ i}} a_{-k}u_{\bar{y}}u - \sum_{\substack{j_L(i)-1 \\ i}}{}' a_{+k}u_y u$$

$$- \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{i_G(j) \\ j}}\Big) ha_{+h}u_x^2 - \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{i_L(j) \\ j}}\Big) ha_{-h}u_{\bar{x}}^2$$

$$- \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{j_G(i) \\ i}}\Big) ka_{+k}u_y^2 - \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{j_L(i) \\ i}}\Big) ka_{-k}u_{\bar{y}}^2$$

$$+ \sum_{\Gamma_h'} (\bar{g}+\delta u)u\Delta\Gamma + \sum_{\Omega_h} S_h\bar{f}u\Big] .$$

Applying the boundary conditions (1.7), (1.10) and (1.11) to the last equation we have

$$\frac{1}{2}\sum_{\Omega_h} S_h u(p\Delta t)^2 + \frac{\Delta t}{2}\sum_{s=1}^{p}\Delta t\sum_{\Omega_h} S_h u_{\bar{t}}^2 \tag{3.7}$$

$$+ \frac{1}{2}\sum_{s=1}^{p}\Delta t\Big[\Big(\sum_{\Omega_h} + \sum_{\substack{i_G(j) \\ j}}\Big) ha_{+h}u_x^2 + \Big(\sum_{\Omega_h} + \sum_{\substack{i_L(j) \\ j}}\Big) ha_{-h}u_{\bar{x}}^2 +$$

$$+ \Big(\sum_{\Omega_h} + \sum_{\substack{j_G(i) \\ i}}\Big) ka_{+k}u_y^2 + \Big(\sum_{\Omega_h} + \sum_{\substack{j_L(i) \\ i}}\Big) ka_{-k}u_{\bar{y}}^2\Big]$$

$$= \frac{1}{2}\sum_{\Omega_h} S_h u(o)^2 + \sum_{s=1}^{p}\Delta t\Big[\sum_{\Gamma_h}\Delta\Gamma(\bar{g}+\delta u)u + \sum_{\Gamma_h'}\Delta\Gamma(\bar{g}+\delta u)u + \sum_{\Omega_h} S_h\bar{f}u\Big]$$

Now we estimate the second sum of the right side by the Schwarz inequality and Lemma 1:

$$\left| \sum_{s=1}^{p} \Delta t \left[\sum_{\Gamma_h} \Delta\Gamma(\bar{g}+\delta u)u + \sum_{\Gamma_h'} \Delta\Gamma(\bar{g}+\delta u)u \right] \right|$$

$$\leqq \sum_{s=1}^{p} \Delta t \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \Delta\Gamma \left[\frac{1}{2} \bar{g}^2 + \left(\frac{1}{2} + |\delta| \right) u^2 \right]$$

$$\leqq \frac{1}{2} \sum_{s=1}^{p} \Delta t \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \bar{g}^2 \Delta\Gamma$$

$$+ \left(\frac{1}{2} + |\delta| \right) \varepsilon \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} \left[h(a_{+h} u_x^2 + a_{-h} u_{\bar{x}}^2) + k(a_{+k} u_y^2 + a_{-k} u_{\bar{y}}^2) \right]$$

$$+ \left(\frac{1}{2} + |\delta| \right) C(\varepsilon) \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h u^2$$

and

$$\left| \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h \bar{f} u \right| \leqq \frac{1}{2} \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h (\bar{f}^2 + u^2) .$$

If we take ε so small that

$$\left(\frac{1}{2} + |\delta| \right) \varepsilon \leqq \frac{1}{4} ,$$

then we have from (3.7) using the above estimates,

$$(3.8) \quad \sum_{\Omega_h} S_h u(p\Delta t)^2 + \frac{1}{2} \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} \left[h(a_{+h} u_x^2 + a_{-h} u_{\bar{x}}^2) + k(a_{+k} u_y^2 + a_{-k} u_{\bar{y}}^2) \right]$$

$$\leqq \sum_{\Omega_h} S_h u(o)^2 + C_1 \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h u^2$$

$$+ \sum_{s=1}^{p} \Delta t \left(\sum_{\Gamma_h} + \sum_{\Gamma_h'} \right) \bar{g}^2 \Delta\Gamma + \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} S_h f^2$$

where $C_1 = 1+(1+2|\delta|)C(\varepsilon)$. Further if we put

$$S(p)=\sum_{s=1}^{p}\Delta t\sum_{\Omega_h} S_h u(s\Delta t)^2$$
$$+\frac{1}{2}\sum_{q=1}^{p}\Delta t\sum_{s=1}^{q}\Delta t\sum_{\Omega_h}\left[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2)+k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2)\right]$$
$$\Psi(p)=\sum_{\Omega_h} S_h u(o)^2 + \sum_{s=1}^{p}\Delta t\left(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\right)\bar{g}^2\Delta\Gamma+\sum_{s=1}^{p}\Delta t\sum_{\Omega_h} S_h\bar{f}^2,$$

we have from (3.8)

$$\text{(3.9)}\qquad \frac{S(p)-S(p-1)}{\Delta t}\leqq C_1 S(p)+\Psi(p).$$

By the same way as in §2 we get

$$S(p)\leqq C_2\Psi(p),$$

and by (3.9) we have consequently

$$\sum_{\Omega_h} S_h u(p\Delta t)^2 + \frac{1}{2}\sum_{s=1}^{p}\Delta t\sum_{\Omega_h}\left[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2)+k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2)\right]$$
$$\text{(3.10)}\qquad \leqq C_3\left[\sum_{\Omega_h} S_h u(o)^2 + \sum_{s=1}^{p}\Delta t\left(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\right)\bar{g}^2\Delta\Gamma+\sum_{s=1}^{p}\Delta t\sum_{\Omega_h} S_h\bar{f}^2\right],$$

where $C_3 = C_1C_2 + 1$. Thus we arrive at

Theorem 3 (Stability) A solution u of the difference problem (3.4), (1.7), (1.10), (1.11) and (3.5) satisfies the energy inequality (3.10) unconditionally for $p\Delta t \leqq T$. (Unconditionaly stable)

Theorem 4 (Convergence) Suppose that the boundary Γ is sufficiently smooth and a solution v of the differential problem has continuous derivatives $\frac{\partial^2 v}{\partial t^2}$, $\frac{\partial^2 v}{\partial x\partial t}$, $\frac{\partial^2 v}{\partial y\partial t}$, $\frac{\partial^3 v}{\partial x^3}$, $\frac{\partial^3 v}{\partial x^2\partial y}$, $\frac{\partial^3 v}{\partial y^2\partial x}$ and $\frac{\partial^3 v}{\partial y^3}$ in $\bar{\Omega}\times[0,T]$.

Then the solution u of our difference scheme converges to the corresponding solution v of the differential problem for Δt, h and $k \to 0$ in the sense of

$$\| v-u \|_h \to 0 \quad \text{uniformly for } p\Delta t \leqq T,$$

where

$$\| w \|_h = \sum_{\Omega_h} S_h w(p\Delta t)^2 + \frac{1}{2} \sum_{s=1}^{p} \Delta t \sum_{\Omega_h} \Big[h(a_{+h} w_x^2 + a_{-h} w_{\bar{x}}^2)$$

$$+ k(a_{+k} w_y^2 + a_{-k} w_{\bar{y}}^2) \Big] .$$

§4 An Elliptic Problem

Here we consider the Neumann problem for an elliptic equation :

$$(4.1) \qquad -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + u = f(x,y) \quad \text{in } \Omega,$$

$$(4.2) \qquad \frac{\partial u}{\partial n} = g(x,y) \qquad \text{on } \Gamma.$$

We know that the above problem has one and only one solution under appropriate smoothness condition on Γ , f and g. Now we can construct a scheme analogous to (1.9) or (3.4) on our net in §1 as follows:

$$(4.3) \qquad -a_{+h}u_x + a_{-h}u_{\bar{x}} - a_{+k}u_y + a_{-k}u_{\bar{y}} - \delta_h\bar{g} \quad + S_h u = S_h\bar{f},$$

with the same boundary condition as (1.7), (1.10) and (1.11) in §1.

We will prove that this difference scheme is uniquely solvable and its solution converges to an exact solution of (4.1) and (4.2). For the purpose we multiply u on both sides of (4.6) and sum over Ω_h, then we have

$$\begin{aligned} &-\sum_{\substack{i_4(j)+1 \\ j}} a_{-h}u_{\bar{x}}u + \sum_{\substack{i_L(j)-1 \\ j}} a_{+h}u_x u - \sum_{\substack{j_4(i)+1 \\ i}} a_{-k}u_{\bar{y}}u + \sum_{\substack{j_L(i)-1 \\ i}} a_{+k}u_y u \\ &+ \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{i_4(j) \\ j}}\Big) h a_{+h}u_x^2 + \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{i_L(j) \\ j}}\Big) h a_{-h}u_{\bar{x}}^2 \\ &+ \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{j_4(i)}\Big) k a_{+k}u_y^2 + \frac{1}{2}\Big(\sum_{\Omega_h} + \sum_{\substack{j_L(i) \\ i}}\Big) k a_{-k}u_{\bar{y}}^2 \\ &- \sum_{\Gamma_h'} \bar{g}u\,\Delta\Gamma + \sum_{\Omega_h} S_h u^2 = \sum_{\Omega_h} S_h\bar{f}u, \end{aligned}$$

(see the case of a heat equation in §3). Applying the boundary conditions (1.7), (1.10) and (1.11), we have

$$\frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{i_G(j)\\ j}}\Big) ha_{+h}u_x^2 + \frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{i_L(j)\\ j}}\Big) ha_{-h}u_{\bar{x}}^2$$

$$+\frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{j_G(i)\\ i}}\Big) ka_{+k}u_y^2 + \frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{j_L(i)\\ i}}\Big) ka_{-k}u_{\bar{y}}^2$$

$$+\sum_{\Omega_h} S_h u^2 = \Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\Delta\Gamma \bar{g}u + \sum_{\Omega_h} S_h \bar{f}u .$$

We can estimate the right side by using the Schwarz's inequality and Lemma 1 as follows :

$$\frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{i_G(j)\\ j}}\Big) ha_{+h}u_x^2 + \frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{i_L(j)\\ j}}\Big) ha_{-h}u_{\bar{x}}^2$$

$$+\frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{j_G(i)\\ i}}\Big) ka_{+k}u_y^2 + \frac{1}{2}\Big(\sum_{\Omega_h}+\sum_{\substack{j_L(i)\\ i}}\Big) ka_{-k}u_{\bar{y}}^2 + \sum_{\Omega_h} S_h u^2$$

$$\leqq \varepsilon_1 \Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big) u^2\Delta\Gamma + \frac{1}{4\varepsilon_1}\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\bar{g}^2\Delta\Gamma$$

$$+\varepsilon \sum_{\Omega_h} S_h u^2 + \frac{1}{4\varepsilon}\sum_{\Omega_h} S_h \bar{f}^2$$

$$\leqq \varepsilon\,\varepsilon_1 \sum_{\Omega_h}\Big[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2) + k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2)\Big]$$

$$+\varepsilon_1(C(\varepsilon)+1)\sum_{\Omega_h} S_h u^2 + \frac{1}{4\varepsilon_1}\Big(\sum_{\Gamma_h}+\sum_{\Gamma_h'}\Big)\bar{g}^2\Delta\Gamma + \frac{1}{4\varepsilon_1}\sum_{\Omega_h} S_h \bar{f}^2,$$

where ε_1 is an arbitrarily small constant, ε and $C(\varepsilon)$ are two constants appearing in Lemmal. We take ε_1 so small that $\varepsilon\,\varepsilon_1 < \frac{1}{4}$ and $\varepsilon_1(C(\varepsilon)+1) < \frac{3}{4}$ hold, then we have from the last inequality,

$$\sum_{\Omega_h}\left[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2)+k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2)\right] + \sum_{\Omega_h} S_h u^2$$
$$\leq \frac{1}{\varepsilon_1}\left(\sum_{\Gamma_h} + \sum_{\Gamma_h'}\right) \bar{g}^2\Delta\Gamma + \frac{1}{\varepsilon_1}\sum_{\Omega_h} S_h\bar{f}^2 .$$

Immediately from the last inequality the uniqueness of the solution of our scheme, and consequently the existence hold. And further we can prove a convergence theorem in the same way as in §2 and §3. Thus

Theorem 5 (Convergence) Suppose that the boundary Γ of Ω is sufficiently smooth and a solution of the problem (4.1) and (4.2) has regularity such that third derivatives are continuous in $\bar{\Omega}$. When h and k tend to zero, a solution of our difference problem converges to the corresponding solution of the differential problem in the sense of

$$\|w\|_h = O(h) \qquad \text{as } h \to o ,$$

where w is the difference between an exact solution and its approximate solution. And

$$\|w\|_h = \sum_{\Omega_h}\left[h(a_{+h}u_x^2 + a_{-h}u_{\bar{x}}^2)+k(a_{+k}u_y^2 + a_{-k}u_{\bar{y}}^2)+S_h u^2\right] .$$

§5 A Numerical Experiment

In order to test our scheme, we consider a simple problem. Let Ω be a unit circle and Γ its boundary. The problem is to find the function v(x,y) satisfying the equations

$$\frac{\partial^2 v}{\partial t^2} - \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) = 0 \qquad \text{in} \quad \Omega \times [o,T] ,$$

$$\frac{\partial v}{\partial n} = 2 \qquad \text{on} \quad \Gamma \times [o,T],$$

$$v(o,x,y) = \frac{\partial v}{\partial t}(o,x,y) = x^2+y^2 \text{ in } \bar{\Omega} .$$

We can easily see that the exact solution takes the form

$$v(t,x,y)=2t^2+x^2+y^2 .$$

For simplicity we take the mesh width $h=k=0.1$ and the time step $t=0.05$ which are considerably coarse. For the calculation we have only to consider an octanant of the unit circle because of symmetry. (See Fig. 2)

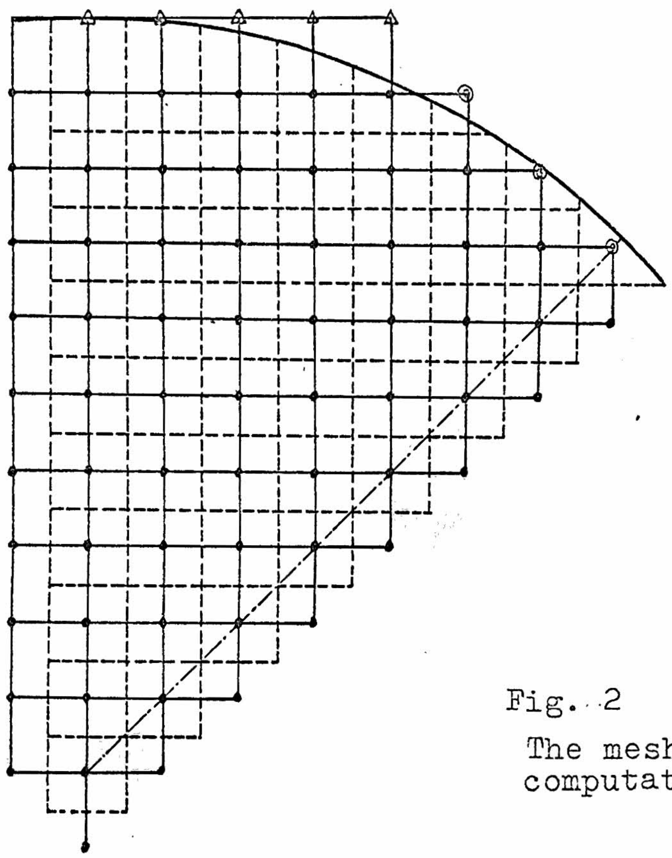

Fig. 2
The mesh points used in computation

The result of the numerical performance shows stable feature of our scheme. We compare the exact solution and our numerical result at $t=10\Delta t=0.5$ and $t=20\Delta t=1.0$

t=0.5 $\frac{\text{Approximation}}{\text{Exact}}$

$\frac{1.46}{1.50}$							
$\frac{1.26}{1.31}$	$\frac{1.27}{1.32}$	$\frac{1.30}{1.35}$	$\frac{1.32}{1.40}$	$\frac{1.37}{1.47}$	$\frac{1.49}{1.56}$		
$\frac{1.09}{1.14}$	$\frac{1.10}{1.15}$	$\frac{1.12}{1.18}$	$\frac{1.14}{1.23}$	$\frac{1.21}{1.30}$	$\frac{1.32}{1.39}$	$\frac{1.43}{1.50}$	
$\frac{0.94}{0.99}$	$\frac{0.95}{1.00}$	$\frac{0.97}{1.03}$	$\frac{1.00}{1.08}$	$\frac{1.05}{1.15}$	$\frac{1.16}{1.24}$	$\frac{1.26}{1.35}$	$\frac{1.36}{1.48}$
$\frac{0.81}{0.86}$	$\frac{0.82}{0.87}$	$\frac{0.84}{0.90}$	$\frac{0.86}{0.95}$	$\frac{0.92}{1.02}$	$\frac{1.06}{1.11}$	$\frac{1.13}{1.22}$	
$\frac{0.70}{0.75}$	$\frac{0.71}{0.76}$	$\frac{0.74}{0.79}$	$\frac{0.77}{0.84}$	$\frac{0.79}{0.91}$	$\frac{0.90}{1.00}$		
$\frac{0.61}{0.66}$	$\frac{0.62}{0.67}$	$\frac{0.65}{0.70}$	$\frac{0.70}{0.75}$	$\frac{0.74}{0.82}$			
$\frac{0.54}{0.59}$	$\frac{0.55}{0.60}$	$\frac{0.58}{0.63}$	$\frac{0.63}{0.68}$				
$\frac{0.49}{0.54}$	$\frac{0.50}{0.55}$	$\frac{0.53}{0.58}$					
$\frac{0.46}{0.51}$	$\frac{0.47}{0.52}$						
$\frac{0.45}{0.50}$							

Table-1

Approximation and exact solutions on the mesh points at t=0.5

t=1.0 $\dfrac{\text{Approximation}}{\text{Exact}}$

$\frac{2.84}{3.00}$							
$\frac{2.64}{2.81}$	$\frac{2.67}{2.82}$	$\frac{2.71}{2.85}$	$\frac{2.77}{2.90}$	$\frac{2.84}{2.97}$	$\frac{2.93}{3.06}$		
$\frac{2.48}{2.64}$	$\frac{2.50}{2.65}$	$\frac{2.53}{2.68}$	$\frac{2.59}{2.73}$	$\frac{2.67}{2.80}$	$\frac{2.75}{2.89}$	$\frac{2.84}{3.00}$	
$\frac{2.33}{2.49}$	$\frac{2.35}{2.50}$	$\frac{2.39}{2.53}$	$\frac{2.45}{2.58}$	$\frac{2.51}{2.65}$	$\frac{2.58}{2.74}$	$\frac{2.66}{2.85}$	$\frac{2.71}{2.98}$
$\frac{2.21}{2.36}$	$\frac{2.23}{2.37}$	$\frac{2.25}{2.40}$	$\frac{2.31}{2.45}$	$\frac{2.39}{2.52}$	$\frac{2.46}{2.61}$	$\frac{2.56}{2.72}$	
$\frac{2.09}{2.25}$	$\frac{2.12}{2.26}$	$\frac{2.16}{2.29}$	$\frac{2.21}{2.34}$	$\frac{2.28}{2.41}$	$\frac{2.36}{2.50}$		
$\frac{2.00}{2.16}$	$\frac{2.03}{2.17}$	$\frac{2.06}{2.20}$	$\frac{2.12}{2.25}$	$\frac{2.21}{2.32}$			
$\frac{1.91}{2.09}$	$\frac{1.96}{2.10}$	$\frac{2.03}{2.13}$	$\frac{2.04}{2.18}$				
$\frac{1.86}{2.04}$	$\frac{1.89}{2.05}$	$\frac{1.97}{2.08}$					
$\frac{1.87}{2.01}$	$\frac{1.87}{2.02}$						
$\frac{1.88}{2.00}$							

Table-2

Approximation and exact solutions on the mesh points at t=1.0

We see that at $t=0.5(t=1.0)$ the maximum error which arise ar to the boundary is $0.12(0.27)$, while the minimum error is 05(0.10), which takes place away from the boundary. These sults shows that total error has order of $O(h)$ in $0 \leqslant t \leqslant 1.0$. ; is the expected matter from our Theorem 2.

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— Note added after the thesis was submitted —

Up to the present the stabibity theory for linear schemes has almost been established, and an excellent review has been written by Lax in the paper [3] (see p.13), which is now indispensable for the study of the subject. In Forward we have extracted sentences from his review, enclosing in quotation marks.